

UNCLASSIFIED

---

AD 259 865

*Reproduced  
by the*

ARMED SERVICES TECHNICAL INFORMATION AGENCY  
ARLINGTON HALL STATION  
ARLINGTON 12, VIRGINIA



---

UNCLASSIFIED

NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

25 9865

CATALOGED BY ASTIA  
AS AD NO. \_\_\_\_\_

An Introduction to Inertial  
Guidance Concepts for  
Ballistic Missiles (A Tutorial Report)

15 MARCH 1961

Prepared by DAVID W. WHITCOMBE

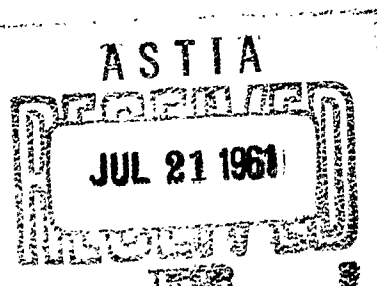
For AIR FORCE BALLISTIC MISSILE DIVISION  
AIR RESEARCH AND DEVELOPMENT COMMAND  
UNITED STATES AIR FORCE, *Inglewood, California*

17 505



AEROSPACE CORPORATION  
CONTRACT NO. AF 04(647)-594

XEROX



**AN INTRODUCTION TO INERTIAL GUIDANCE  
CONCEPTS FOR BALLISTIC MISSILES  
(A TUTORIAL REPORT)**

by  
David W. Whitcombe

**AEROSPACE CORPORATION  
El Segundo, California**

**Contract AF 04(647)-594**

**15 March 1961**

**Prepared for  
AIR FORCE BALLISTIC MISSILE DIVISION  
AIR RESEARCH AND DEVELOPMENT COMMAND  
UNITED STATES AIR FORCE  
Inglewood, California**

## ABSTRACT

Various aspects of guidance and control for ballistic missiles are presented. A discussion of the required velocity vector is given with several examples based upon the flat earth model. Error analysis procedures involving inertial reference and measurement systems are treated in some detail.

Missile steering techniques are approached in such a way that stability is not a problem. Procedures for insuring that the missile velocity will be controlled to the required velocity without payload loss penalties are given. This is accomplished by introducing trajectory-shaping functions that cause the guided ballistic missile to closely follow the nominal reference trajectory.

The steering discussion initially assumes a point mass missile. Control system complications that result from actual missile dynamics are treated separately.

## CONTENTS

	Page
I. GENERAL DISCUSSION. . . . .	1
II. INERTIAL COORDINATE SYSTEMS . . . . .	3
III. INERTIAL GUIDANCE MEASUREMENT ERRORS. . . . .	5
IV. POSITION AND VELOCITY DETERMINATION . . . . .	14
V. THE REQUIRED VELOCITY VECTOR. . . . .	19
VI. BALLISTIC MISSILE STEERING METHODS. . . . .	29
VII. BALLISTIC MISSILE CONTROL SYSTEMS . . . . .	43
REFERENCES . . . . .	51
APPENDIX A: Derivation of the Force Equation for Ballistic Missiles. . . . .	52
APPENDIX B: The Gravity Turn Pitch Program . . . . .	54

## I. GENERAL DISCUSSION

The purpose of a ballistic missile guidance system is to obtain engine steering and shutoff information by operating on measured data. These systems are basically of two types, radio and inertial. A summary description of each is given below.

Radio guidance makes position and/or velocity measurements from ground-based radar. These measurements are input to a ground-based computer where guidance commands are computed. These commands are sent to the missile control system via a radio link. A radio guidance system requires missile visibility from the tracking radar site. In some cases a downrange station must be provided to insure this visibility. A constraint on the missile attitude may also be imposed because the beacon antenna is attached to the missile. The accuracy of the all-electronic radio guidance systems, especially those with long base lines, is generally better than may be obtained with inertial guidance systems (IGS's).

An all-inertial guidance system for ballistic missiles makes acceleration measurements in a gyro-stabilized platform coordinate system.\* These measurements are input to an airborne computer where guidance commands are computed and relayed to the control system. The control system used with an IGS may be simplified in that the conventional gyro attitude reference normally used with radio guidance may be replaced by the inertial platform. The electromechanical sensors used in inertial guidance systems are subject to error. The precision gyros, used in the platform, drift over long periods of time resulting in significant coordinate system errors. The bias and scale factor errors in the accelerometer result in position and velocity errors in the navigation loop. The above IGS errors are offset by the flexibility offered by a self-contained non-radiating system. Present day IGS accuracy is satisfactory for most

---

\* Guidance principles that are directly related to strapped-down IG systems will not be discussed in this report.

space applications in the earth's gravitational field; this system may also be used for ballistic missile launch guidance to lunar impact. For more complicated missions, lasting many days or months, an IGS may be aided with a star and/or planet tracker. Radio guidance may also be used as an inertial aid, especially when target acquisition greatly simplifies the guidance task.

The principal differences between radio and all-inertial systems then involve the form of the original measurements. With either system sufficient data must be measured to allow the calculation of position and velocity vectors,  $\vec{R}$  and  $\vec{V}$ , of the missile in some suitable coordinate system. A velocity deviation measured from the desired missile velocity may be obtained. The thrust acceleration,  $\vec{a}_T$ , may then be steered in direction so as to drive the yaw and pitch components of the velocity deviation to zero. When the axial (or roll) error velocity component is reduced to zero, engine thrust is terminated. The missile will then coast to its target under the influence of mass attraction forces exerted by the earth, sun, other planets, etc.

Note that it is assumed here that the thrust acceleration,  $\vec{a}_T$ , can only be steered in direction and not controlled in magnitude. This lack of control does not impose a serious restraint because the magnitude of  $\vec{a}_T$  is designed to be reproducible within tolerances of a few percent.

This memo will be concerned with all-inertial guidance systems used for ballistic missile guidance. The duration of powered flight in a ballistic trajectory is usually several minutes, during which accelerations may reach 15 g's. All guidance equipment on board the missile must operate with a high degree of reliability in this environment.

The following sections of this memo will discuss inertial coordinate systems, measurement errors, the navigation loop, guidance computations, and ballistic missile steering.



## II. INERTIAL COORDINATE SYSTEMS

A coordinate system (CS) will be specified in this section in terms of three mutually orthogonal inertial directions. These directions are indicated by three unit vectors, which are termed basis vectors. Each coordinate system is then defined by specifying the directions of the three basis vectors. Four coordinate systems will be discussed. The orientation of the three basis vectors will in each case be apparent. In no case will the transformation from one CS to another be written out as a set of equations. The orientation of the basis vectors is always assumed known, and these transformations can generally be written by inspection in terms of the geometry of the system.

The inertial measurement unit consists of three accelerometers mounted on a gyro-stabilized platform. In this memo, it will be assumed that the reference platform will maintain a fixed orientation with respect to inertial space. Platform hardware such as gimbals, the porro prism, etc., must be used to specify three orthogonal platform directions. These directions define the platform coordinate system. Initial platform CS alignment is accomplished with respect to an external CS. The three input axes of the platform stabilizing gyros define a second coordinate system. This CS must be defined in order to perform platform drift error analyses. It will be shown in the next section that the orientation of each gyro must be specified. The three accelerometers are fixed rigidly to the inertial platform. The directions of the three accelerometers define the accelerometer coordinate system. This is an important CS because it is the one in which the measurements of sensed acceleration,  $\vec{a}_T$ , are made.

A fourth coordinate system is of interest. This will be referred to as the computing coordinate system, and is the one used for guidance calculations. This CS may be an earth-centered inertial CS in which the expression for gravity may be easily calculated. Other coordinate systems have other desirable properties. The four coordinate systems above may all be equivalent or they may all be different depending on actual hardware and

guidance mechanizations. In any case, the transformations from any one CS to any other CS will be known. Hence, for purposes of this memo, it may be assumed that the accelerometer coordinate system and the computing coordinate system are identical.

The computing CS will be specified in terms of the three mutually orthogonal unit vectors  $\vec{i}$ ,  $\vec{j}$ , and  $\vec{k}$ . The missile position and velocity may be written in terms of this set of inertial basis vectors as

$$\begin{aligned}\vec{R} &= X \vec{i} + Y \vec{j} + Z \vec{k} \\ \dot{\vec{R}} &= \dot{X} \vec{i} + \dot{Y} \vec{j} + \dot{Z} \vec{k}\end{aligned}$$

where  $X, Y, Z$  and  $\dot{X}, \dot{Y}, \dot{Z}$  are the components of missile position and velocity respectively. The computing coordinate system inertial basis vectors  $\vec{i}, \vec{j}, \vec{k}$  may be used to evaluate the components of any vector used in the following section. For example,

$$\begin{aligned}X &= \vec{R} \cdot \vec{i} \\ Y &= \vec{R} \cdot \vec{j} \\ Z &= \vec{R} \cdot \vec{k}\end{aligned}$$

### III. INERTIAL GUIDANCE MEASUREMENT ERRORS

Inertial guidance measurement errors affect both the magnitude and direction of the sensed acceleration. These error sources are conveniently classed as follows:

- a) Accelerometer bias, scale factor, and second-degree errors;
- b) Accelerometer and platform misalignments;
- c) Fixed and acceleration-dependent gyro drift rates.

Errors in Class (a) affect the magnitude of the sensed acceleration, while Class (b) and (c) errors affect the direction.

Detailed discussions of inertial guidance error analysis techniques are given in References (1) and (2). A summary description is given below.

#### Sensed Acceleration Magnitude Errors:

All accelerometers contain some moving parts. When acceleration is sensed, forces within the instrument cause an internal displacement that is proportional to the acceleration. When the device is used during ballistic coast periods, no such forces act on the internal parts of the instrument. There are no internal displacements in a zero- $\underline{g}$  field when changes in  $\underline{g}$  within the instrument are neglected. Hence, the reading of an ideal accelerometer is

$$\underline{\ddot{a}}_T = \underline{\ddot{R}} - \underline{\ddot{g}} \quad (3.1)$$

where

- $\underline{\ddot{R}}$  = acceleration of the missile with respect to inertial space
- $\underline{\ddot{g}}$  = acceleration of gravity
- $\underline{\ddot{a}}_T$  = the sum of all external accelerations acting on the missile except  $\underline{\ddot{g}}$

Let  $a_{Tx}$  and  $a'_{Tx}$  denote the ideal and actual readings of the  $\vec{i}$  accelerometer. The accelerometer error  $\Delta a_{Tx}$  is then

$$\Delta a_{Tx} = a'_{Tx} - a_{Tx}$$

Accelerometer measurement errors result from bias, cross acceleration, and nonlinearity errors. A general expression for accelerometer error is given as

$$\Delta a_{Tx} = k_0 + k_1 a_{Tx} + k_2 a_{Ty} + k_3 a_{Tz} + k_{11} a_{Tx}^2 + k_{12} a_{Tx} a_{Ty} + \dots \quad (3.2)$$

where

$k_0$  = bias coefficient

$k_1$  = scale factor coefficient

$k_2, k_3$  = cross axis coefficients

$k_{11}$  = second-degree error coefficient

$k_{12}$ , etc. = additional nonlinearity error coefficients

$a_{Tx}, a_{Ty}, a_{Tz}$  = components of sensed acceleration

Expressions similar to equation (3.2) are used for the  $\vec{j}$  and  $\vec{k}$  accelerometers. The relative magnitude of the error coefficients in equation (3.2) depends on the particular accelerometer design. The actual values are usually obtained from the manufacturer or impartial testing agencies.

#### Accelerometer and Platform Misalignments:

The accelerometers cannot be mounted on the platform without introducing some error. Similarly the platform cannot be leveled and aligned in azimuth without error. As a result, six misalignment angles, two for each accelerometer, are required to express the misalignment of the accelerometer package with respect to inertial space.

Let  $\vec{i}, \vec{j}, \vec{k}$  denote the desired orientation of the three accelerometers. The actual orientations  $\vec{i}', \vec{j}',$  and  $\vec{k}'$  are then

$$\begin{aligned}\vec{i}' &= \vec{i} + \phi_{12}\vec{j} + \phi_{13}\vec{k} \\ \vec{j}' &= \vec{j} + \phi_{23}\vec{k} + \phi_{21}\vec{i} \\ \vec{k}' &= \vec{k} + \phi_{31}\vec{i} + \phi_{32}\vec{j}\end{aligned}\tag{3.3}$$

where the  $\phi_{12}, \dots, \phi_{32}$  denote the small misalignment angles. For example,  $\phi_{23}$  is the misalignment of the  $\vec{j}$  accelerometer in the  $\vec{k}$  direction. The misalignment angles may be specified in several ways. For example, three angles may be used to specify the platform level and azimuth alignment. These angles are equivalent to  $\phi_{13}, \phi_{23},$  and  $\phi_{12}$ . The  $\vec{i}$  accelerometer may then be mounted on the platform with zero error assumed. The  $\vec{j}$  accelerometer is then mounted on the platform with respect to  $\vec{i}$ ; the error is  $\phi_{21}$ . The  $\vec{k}$  accelerometer is then mounted with errors  $\phi_{31}$  and  $\phi_{32}$ . Again, let  $a_{Tx}, a_{Ty},$  and  $a_{Tz}$  denote the ideal accelerometer measurements. The actual measurements  $a'_{Tx}, a'_{Ty}, a'_{Tz},$  are obtained from equation (3.3) as

$$\begin{aligned}a'_{Tx} &= a_{Tx} + \phi_{12}a_{Ty} + \phi_{13}a_{Tz} \\ a'_{Ty} &= a_{Ty} + \phi_{23}a_{Tz} + \phi_{21}a_{Tx} \\ a'_{Tz} &= a_{Tz} + \phi_{31}a_{Tx} + \phi_{32}a_{Ty}\end{aligned}\tag{3.4}$$

The last two terms in each of equations (3.4) are error terms. These are easily calculated using standard missile simulation programs.

#### Platform Drift Error:

Coordinate system errors result from drifts in the platform stabilizing gyros. In addition to the steady drift rate of the gyro, acceleration-dependent drift rates may exist. These latter drift rates are referred to as mass unbalance and anisoelastic drift rates. The following discussion is included to aid in understanding the nature of IGS platform drift.

A platform uses three single-degree-of-freedom gyros or two two-degree-of-freedom gyros. The three orthogonal axes of a single-degree-of-freedom gyro are shown in Figure 3.1.

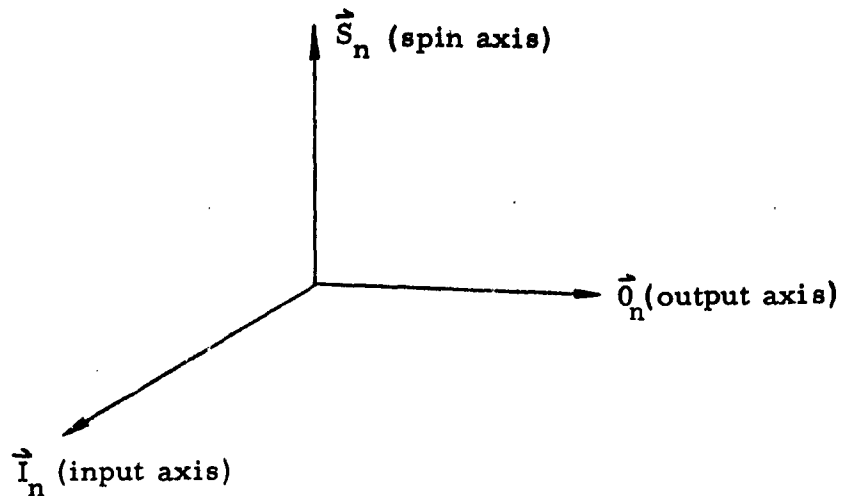


Figure 3.1 - Axes of a SDF Gyro

In Figure 3.1, the subscript  $n = 1, 2, \text{ or } 3$  is added to  $\vec{S}$ ,  $\vec{I}$ , and  $\vec{O}$  to indicate the spin, input, or output axis of the  $n^{\text{th}}$  platform gyro. A platform SDF rate integrating gyro is designed to measure only angular rates about the input axis. An angle pickoff on the output axis measures the input angular rate. Angular rates about the output axis do not cause a pickoff indication because the viscous fluid damping constrains the case motion to follow the inner gimbal motion.

A conventional two-degree-of-freedom gyro differs from a SDF gyro in that an additional gimbal and pickoff are added. As such, the 2-DF gyro

may be considered equivalent in performance to two SDF gyros. When two 2-DF gyros are used to stabilize the platform a redundant control direction is provided. This redundant direction will be ignored in this analysis. The subscript  $n$  will be used to denote any one of three orthogonal input axes that are used.

A 2-DF gyro does not require the viscous damping restraint. In fact, the flotation fluid has negligible damping. The additional gimbal and pickoff is provided so that angular displacements about both input axes may be measured. The directions of these axes are shown in Figure 3.2. Note that each input axis is also an output axis.

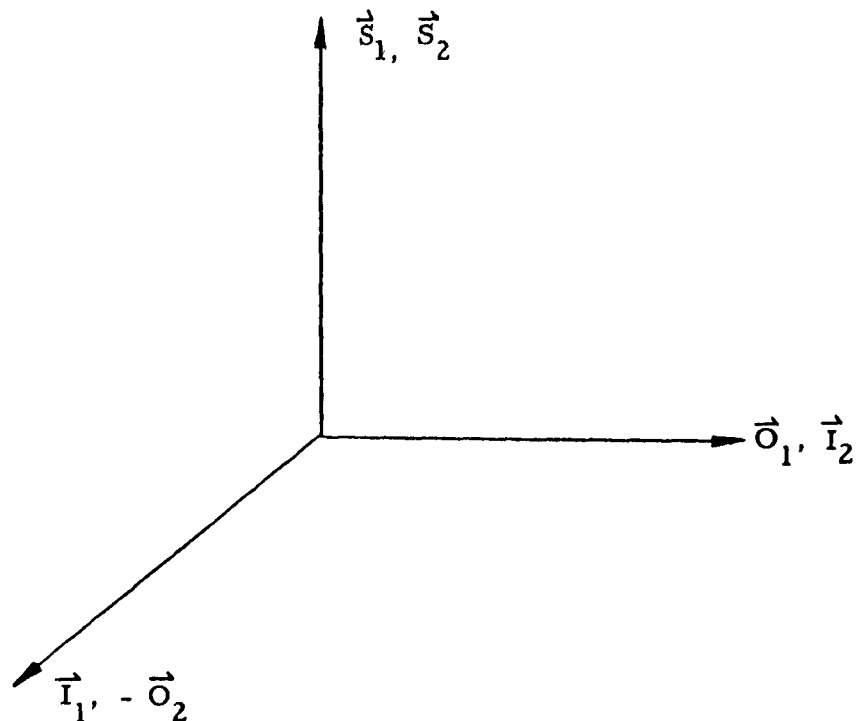


Figure 3.2 - Axes for a 2-DF Gyro

The angular momentum of the gyro spin rotor is large compared to the gimbal momenta and certain other dynamical terms discussed in

References (3) and (4). Hence, the law of the gyro is approximated as

$$\vec{T}_n = \vec{\omega}_n \wedge \vec{H}_n \quad (3.5)$$

where

$\vec{H}_n$  = vector angular momentum of the gyro wheel

$\vec{T}_n$  = residual unbalance torques within the gyro

The symbol  $\wedge$  is used to denote the vector cross product.

Both SDF and 2-DF gyros drift as a result of unbalance torques acting about the output axis. When both sides of equation (3.5) are multiplied by  $\vec{O}_n$  (scalar product) the result is

$$\begin{aligned} \vec{T}_n \cdot \vec{O}_n &= (\vec{\omega}_n \wedge \vec{H}_n) \cdot \vec{O}_n \\ &= H_n \vec{\omega}_n \cdot (\vec{S}_n \wedge \vec{O}_n) \\ &= -H_n \vec{\omega}_n \cdot \vec{I}_n \end{aligned}$$

Hence, the platform drift rates about the three input axes,  $\omega_n$ , may be written as\*

$$\omega_n = -\frac{1}{H_n} \vec{T}_n \cdot \vec{O}_n \quad (3.6)$$

Fixed, i.e., constant or steady, gyro drift rates are usually denoted by R. These drift rates result from fixed residual torques,  $\vec{T}_o$ , within the gyro such as bearing frictions and pickoff reactions. Then the fixed drift rate of the  $n^{\text{th}}$  platform gyro is

$$R_n = -\frac{1}{H_n} \vec{T}_{on} \cdot \vec{O}_n \quad (3.7)$$

---

\*The presentation in this section is based on original work by T. W. Layton and H. Cohen.



A mass unbalance within the  $n^{\text{th}}$  gyro is approximately equivalent to the torque

$$\vec{T}_n = m_n \vec{\delta}_n \wedge \vec{a}_T \quad (3.8)$$

where

$m_n$  = internal gyro mass

$\vec{\delta}_n$  = vector displacement of the center of mass of the rotor gimbal system

$\vec{a}_T$  = sensed acceleration vector

The quantities  $\vec{\delta}_n$  and  $\vec{a}_T$  may be written as

$$\begin{aligned} \vec{a}_T &= (\vec{a}_T \cdot \vec{O}_n) \vec{O}_n + (\vec{a}_T \cdot \vec{I}_n) \vec{I}_n + (\vec{a}_T \cdot \vec{S}_n) \vec{S}_n \\ \vec{\delta}_n &= (\vec{\delta}_n \cdot \vec{O}_n) \vec{O}_n + (\vec{\delta}_n \cdot \vec{I}_n) \vec{I}_n + (\vec{\delta}_n \cdot \vec{S}_n) \vec{S}_n \end{aligned}$$

Hence, the  $n^{\text{th}}$  gyro drift rate resulting from mass unbalance is

$$\omega_{U_n} = U_{I_n} (\vec{a}_T \cdot \vec{S}_n) - U_{S_n} (\vec{a}_T \cdot \vec{I}_n) \quad (3.9)$$

where

$$U_{I_n} = \frac{m_n}{H_n} (\vec{\delta}_n \cdot \vec{I}_n) \quad (3.9A)$$

$$U_{S_n} = \frac{m_n}{H_n} (\vec{\delta}_n \cdot \vec{S}_n) \quad (3.9B)$$

Anisoelastic gyro drift rate results when the gyro is placed in an acceleration field. In this environment the gyro will deform. The resulting mass shift is obtained as  $m_n \vec{\epsilon}_n$ , where

$$\vec{\epsilon}_n \cong K_{O_n} (\vec{a}_T \cdot \vec{O}_n) \vec{O}_n + K_{I_n} (\vec{a}_T \cdot \vec{I}_n) \vec{I}_n + K_{S_n} (\vec{a}_T \cdot \vec{S}_n) \vec{S}_n \quad (3.10)$$

The constants  $K_{On}$ ,  $K_{In}$ ,  $K_{Sn}$  are elastic coefficients of the  $n^{\text{th}}$  platform gyro. The resulting residual torque is

$$\vec{T}_n = m_n \vec{e}_n \wedge \vec{a}_T \quad (3.11)$$

Substituting equations (3.11) and (3.10) into (3.6) gives

$$\omega_{A_n} = 2K_n (\vec{a}_T \cdot \vec{I}_n) (\vec{a}_T \cdot \vec{S}_n) \quad (3.12)$$

where

$$2K_n = \frac{K_{In} - K_{Sn}}{H_n} \quad (3.12A)$$

$K_n$  = anisoelastic drift rate coefficient  
for the  $n^{\text{th}}$  platform gyro

The gyro drift rate,  $\omega_{A_n}$ , is referred to as the anisoelastic drift rate of the  $n^{\text{th}}$  platform stabilization gyro. The gyro is said to be isoelastic if  $K_{In} = K_{Sn}$ . This gyro design goal has not yet been achieved.

The resultant gyro drift rate is denoted by  $\sigma_D$ . An expression for  $\sigma_D$  is obtained by adding the fixed, mass unbalance and anisoelastic drift rates. The result, for the  $n^{\text{th}}$  platform gyro, is

$$\sigma_D = R_n + U_{In} a_{Sn} - U_{In} a_{In} + 2K_n a_{In} a_{Sn} \quad (3.13)$$

where

$$a_{Sn} = \vec{a}_T \cdot \vec{S}_n \quad (3.13A)$$

$$a_{In} = \vec{a}_T \cdot \vec{I}_n \quad (3.13B)$$

The above discussion applies to conventional ball-bearing floated platform gyros. Gyros utilizing other designs may have internal acceleration-dependent torques that are not calculated above.

There are many ways that three SDF gyros may be oriented on the platform. One such orientation is given in Table 3.1, where it is assumed that the thrust acceleration is always contained in the X-Z plane.

	Gyro 1	Gyro 2	Gyro 3
X Direction	$\vec{I}_1$	$\vec{S}_2 \cos \alpha + \vec{O}_2 \sin \alpha$	$\vec{O}_3$
Y Direction	$\vec{S}_1$	$\vec{I}_2$	$\vec{S}_3$
Z Direction	$-\vec{O}_1$	$-\vec{S}_2 \sin \alpha + \vec{O}_2 \cos \alpha$	$\vec{I}_3$

Table 3.1 - An Optimum Gyro Orientation

The following features of Table 3.1 should be noted:

- The input axes are orthogonal and define the gyro coordinate system.
- The SDF gyros 1 and 3 could be replaced by a single 2-DF gyro.
- Since  $\vec{a}_T$  is in the X-Z plane, there is no anisoelastic drift rate regardless of the value of the angle  $\alpha$ .
- The platform could be rotated about the  $\vec{j}$  direction through an angle  $\beta$  without introducing anisoelastic drift rates.

The gyro orientation given in Table 3.1 successfully eliminates anisoelastic drift rates, but not mass unbalance drift rates. It is possible to choose the angles  $\alpha$  and  $\beta$  to minimize these latter drift rates. When this has been done the gyro orientation is said to be optimized.

#### IV. POSITION AND VELOCITY DETERMINATION

A ballistic missile guidance system does not always require explicit information as to position and velocity, as in the case of the Q-guidance method proposed by MIT. The guidance methods to be discussed in this memo, however, require that the inertial position vector,  $\vec{R}$ , and the inertial velocity vector,  $\dot{\vec{R}}$ , be determined. The equation to be solved is

$$\ddot{\vec{R}}(t) = \vec{g}(\vec{R}) + \vec{a}_T(t) \quad (4.1)$$

where

$\vec{g}$  = gravitational acceleration  
 $\vec{a}_T$  = sensed acceleration (measured by the accelerometers)  
 $\ddot{\vec{R}}$  = inertial acceleration

Equation (4.1) is integrated in the IGS computer. The block diagram shown in Figure 4.1 indicates the steps in the solution. The following notation is

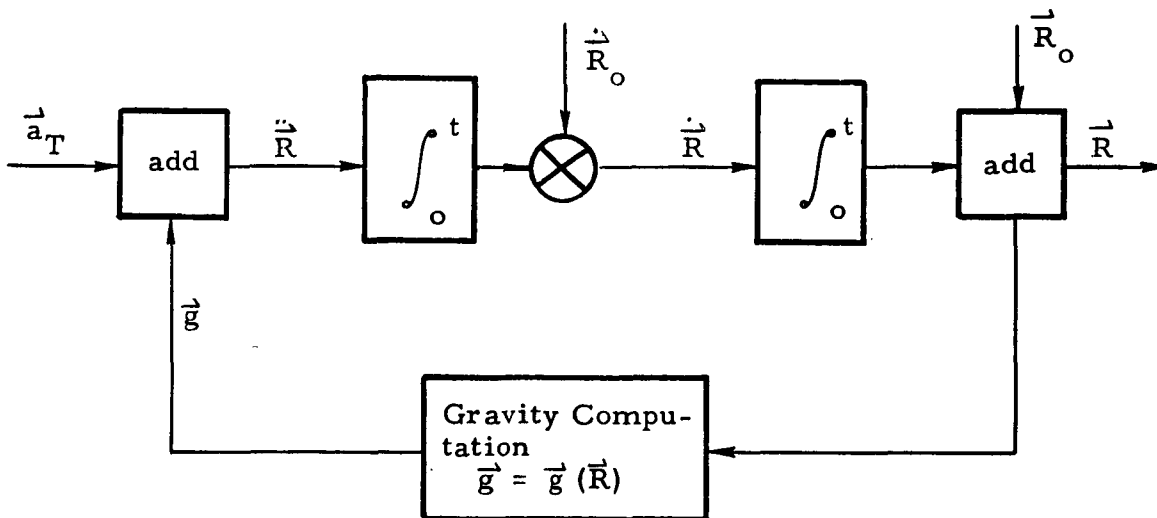


Figure 4.1 - Position and Velocity Determination

used in Figure 4.1

$\vec{R}_0$  = initial position vector of missile

$\dot{\vec{R}}_0$  = initial velocity vector of missile

Equation (4.1) is solved in the computing coordinate system. If the accelerometer coordinate system is not the same CS, a transformation of  $\vec{a}_T$  to the computing CS must be included.

Division and square root operations are slow processes for many IGS computers. Hence, the direct computation of  $\vec{g}$  is a slow process when  $\vec{g}$  has the form

$$\vec{g} = - \frac{GM}{r^3} \vec{R} \quad (4.2)$$

where

$\vec{R}$  = missile position vector measured from the center of the earth

$$r = \sqrt{\vec{R} \cdot \vec{R}} = \sqrt{X^2 + Y^2 + Z^2} = |\vec{R}|$$

In equation (4.2) the position vector of the missile,  $\vec{R}$ , is expressed in a coordinate system with origin at the center of the earth. At worst this requires a simple transformation to the computing CS. The gravity function may be expanded into a Taylor's series of the form

$$\vec{g} = - (\vec{R} + \vec{R}_0) F(X, Y, Z) \quad (4.3)$$

where

$$F = C_0 + C_1 X + C_2 Z + C_4 X^2 + C_5 Z^2 + C_6 XZ$$

A complete description of this expansion is given in Reference (5).

In equation (4.3) it is assumed that the nominal thrust acceleration vector is always contained in the X - Z plane. The above computational form for  $\vec{g}$  includes oblateness terms while the form given in equation (4.2) does not. The coefficients,  $C_1$ , required by equation (4.3) are obtained by Taylor's series expansion about a downrange expansion point. Other methods for mechanizing the gravity computation exist. For example, equation (4.2) may be calculated using difference equations. In some cases, for a short high-altitude burn it will suffice to use a constant vector for gravitational acceleration.

Most of the accelerometers currently used in IG systems are integrating accelerometers. That is, the output of the accelerometer is thrust velocity,  $\vec{V}_T$ , where

$$\vec{V}_T = \int_0^t \vec{a}_T dt$$

When this is the case the navigation loop shown in Figure 4.2 may be used.

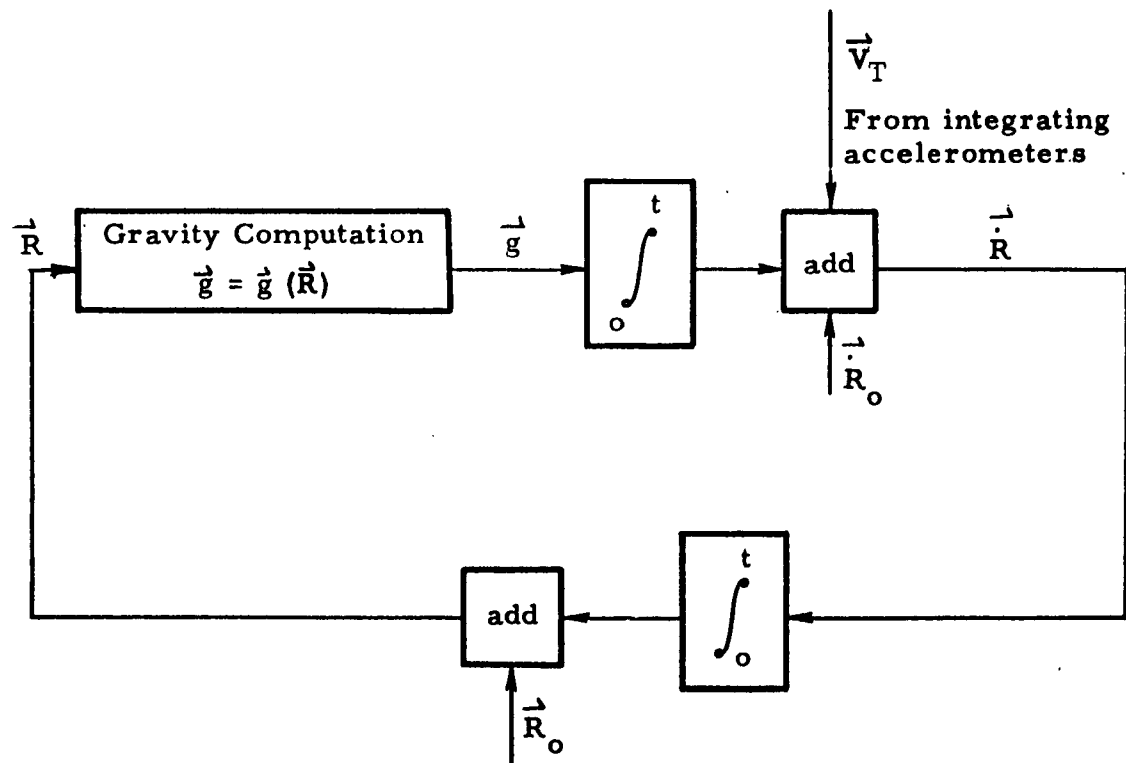


Figure 4.2 - A Possible Navigation Loop When an Integrating Accelerometer Is Used.

The initial position and velocity  $\vec{R}_0$  and  $\dot{\vec{R}}_0$ , required in Figures 4.1 and 4.2 are readily determined when the missile is launched from the earth. In some space missions a second burning period may be required after the coast (ballistic free flight) phase. The determination of initial conditions for this second burn may be made in two ways. Since  $\vec{g}$  is the only external acceleration acting on the missile during coast, it is only necessary to integrate  $\vec{g}$  (twice) during the coast period. When this method is used the accelerometer inputs in Figures 4.1 and 4.2 are opened so that accelerometer bias errors are not introduced.

The second method requires that the six initial conditions at the start of the second burn be predicted using first-burnout position and velocity information. It is also necessary to predict the time of free flight,  $t_{ff}$ , in order to provide an engine start command for the second burn. It is shown in Section V that at first burnout, the missile will satisfy four guidance restraints at the start-point of the second burn. This start-point is the "target" for the first burn. When the four guidance restraints are  $X = X_T$ ,  $Y = Y_T$ ,  $Z = Z_T$ ,  $\dot{Z} = \dot{Z}_T$ , then three quantities remain to be determined. These may be predicted using expressions of the following form:

$$\dot{X}_T = \dot{X}_{TN} + C_{11}\Delta X + C_{12}\Delta Y + C_{13}\Delta Z + C_{14}\Delta t$$

$$\dot{Y}_T = \dot{Y}_{TN} + C_{21}\Delta X + C_{22}\Delta Y + C_{23}\Delta Z + C_{24}\Delta t$$

$$t_{ff} = t_{ffn} + C_{41}\Delta X + C_{42}\Delta Y + C_{43}\Delta Z + C_{44}\Delta t$$

where

$$\Delta X = X - X_N$$

$$\Delta Y = Y - Y_N$$

$$\Delta Z = Z - Z_N$$

$$\Delta t = t - t_N$$

$X_N, Y_N, Z_N, t_N$  = nominal position and time at first burnout

$X, Y, Z, t$  = actual position and time at first burnout

$\dot{X}_{TN}, \dot{Y}_{TN}$  = nominal velocity components at the target

$t_{ff}$  = nominal time of free flight

When the above method is used the guidance computer (except the clock) as well as the accelerometer and accelerometer electronics may be shut off. This results in a saving of power, and therefore weight.



## V. THE REQUIRED VELOCITY VECTOR

A ballistic missile engine whose thrust may be controlled in magnitude as well as direction is said to have thrust magnitude control. Ballistic missiles with this feature can be made to fly the nominal desired time-specified trajectory. That is, such missiles could be controlled to burn-out at a prespecified position, velocity, and time.

It will be assumed in this memo that the ballistic missile engines used do not have thrust magnitude control. However, it will be assumed that the thrust and mass magnitudes are reproducible within a few percent. These ballistic missiles can then be steered only by pitching and yawing the thrust vector. With this type of control it is possible to steer the missile velocity vector to the value specified by some function of missile position,  $\vec{R}$ , and time,  $t$ . This desired velocity function will be termed the required velocity vector,  $\vec{V}_R = \vec{V}_R(\vec{R}, t)$ . This vector function is obtained on the limiting assumption that an impulsive thrust engine is used. That is, at any time during the powered flight, it is assumed that a velocity increment,  $\Delta\vec{V}$ , may be added such that

$$\dot{\vec{R}} + \Delta\vec{V} = \vec{V}_R$$

This is referred to as a limiting assumption because additional guidance constraints may be imposed when the actual finite thrusting capabilities of the missile are used in the analysis.

The limiting assumption will be made in the analysis that follows. Then the specification of four general guidance constraints is necessary in order to uniquely determine  $\vec{V}_R(\vec{R}, t)$ . A consideration of Figure 5.1 will help to clarify this matter.

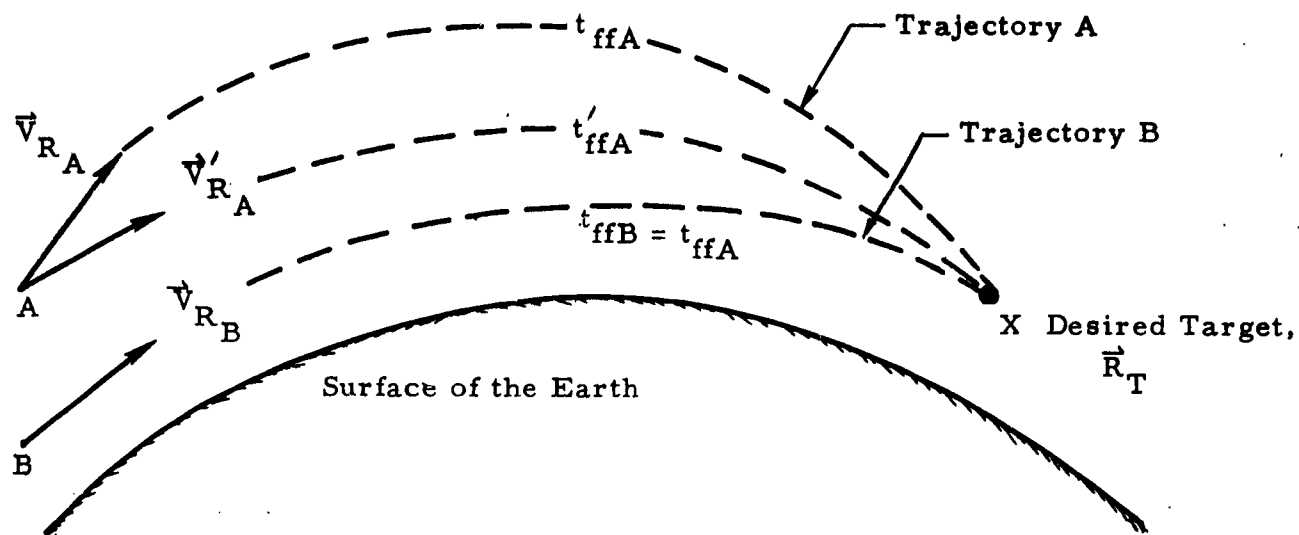


Figure 5.1 - Three Coast Trajectories, All Hitting the Desired Target

The three free-flight trajectories shown in Figure 5.1 all "hit" the desired target. That is, after a period of time,  $t_{ff}$ , each of the three trajectories satisfies the three guidance constraints that

$$X = X_T$$

$$Y = Y_T$$

$$Z = Z_T$$

Consider the two trajectories originating at point A. In order to specify the required velocity vector at point A, an additional guidance constraint

must be imposed. For example, it may be required to hold the time of free flight to a constant value, as is done in trajectories A and B. Equally well, any one of the following restraints could be prespecified:

- a) Burnout velocity magnitude
- b) Burnout vertical velocity
- c) Burnout velocity vector elevation angle
- d) Burnout energy
- e) Burnout angular momentum magnitude
- f) Velocity magnitude at the target
- g) Vertical velocity at the target
- h) Total time of flight from liftoff to the target

In addition to the above, many more restraints which are desirable could be listed.

It is not necessary to include hitting the target as three restraints. A required velocity vector field,  $\vec{V}_R(\vec{R}, t)$ , can be found that will cause the free-flight velocity vector to satisfy the following four conditions:

- 1)  $X = X_T$
- 2)  $Z = Z_T$
- 3)  $\dot{Y} = \dot{Y}_T$
- 4) Constant total time of flight from liftoff

The  $t_{ff}$ ,  $\dot{X}_T$ ,  $\dot{Z}_T$ , and  $Y_T$  will all vary in the preceding example depending on the burnout time and position vector. Another interesting example will be included. It is possible to steer the missile so that the velocity vector at burnout has some desired value. In this case the four guidance restraints are the following:

$$1) \quad \dot{X} = \dot{X}_T$$

$$2) \quad \dot{Y} = \dot{Y}_T$$

$$3) \quad \dot{Z} = \dot{Z}_T$$

$$4) \quad t_{ff} = 0$$

In this example the burnout position and time will vary depending on the off nominal performance of the propulsion system, etc. The above examples serve to illustrate that all essential guidance information is included in the specification of the required velocity vector,  $\vec{V}_R$ . This follows from the fact that all missiles can be controlled (theoretically) such that they will have exactly the required velocity vector at burnout. They will therefore satisfy the prescribed guidance restraints. Any two missiles that burn out at the same vector position and time will have identical free-flight trajectories. Note that the missile must be steered in pitch in order to achieve  $\vec{V}_R = \vec{V}_R(\vec{R}, t)$ .

The required velocity vector is sometimes discussed with a meaning different from that described above. For example, the four guidance restraints may be the following:

$$1, 2, 3) \quad \vec{R} = \vec{R}_T$$

$$4) \quad \text{Arbitrary vertical velocity, } \dot{Z}, \text{ at burnout.}$$

Note that the fourth restraint, above, is quite different from condition b) previously discussed, where it was required that the burnout vertical velocity be steered to a prespecified constant. Condition 4) is used whenever it is required to pitch the missile according to some preassigned pitch program. In this case the missile vertical velocity,  $\dot{Z}$ , will vary according to missile performance. The variation in  $\dot{Z}$  may be indicated by writing the guidance control function as:

$$\vec{V}_R = \vec{V}_R(\vec{R}, \dot{Z}, t) \quad \text{where } \dot{Z} \text{ is arbitrary.}$$

Using  $\vec{V}_R$  in this form leads to satisfactory performance for missiles that are designed to impact on the earth. Note that only three guidance restraints may now be imposed. That is, at  $\vec{R} = \vec{R}_T$  along the coast trajectory, the coast time of free flight will necessarily vary with missile performance. This memo will not further discuss guidance systems using  $\vec{V}_R = \vec{V}_R(\vec{R}, \dot{Z}, t)$ . Only guidance systems using  $\vec{V}_R = \vec{V}_R(\vec{R}, t)$ , which require pitch steering, will be discussed.

Some simple examples dealing with required velocity vector calculation will now be given. The calculation of  $\vec{V}_R$  involves only the free-flight equations of motion. On a flat earth these may be written as

$$\ddot{\vec{R}} = \vec{g} \quad (5.1)$$

where  $\vec{g}$  is a constant. The solution is

$$\vec{R} = \vec{R}_0 + \dot{\vec{R}}_0 t + \frac{1}{2} \vec{g} t^2 \quad (5.2)$$

In equation (5.2),  $\vec{R}_0$  and  $\dot{\vec{R}}_0$  denote the burnout conditions. It is now required to find the burnout velocity that will cause the missile to coast to the target,  $\vec{R}_T(t)$ , after a time of free flight,  $t_{ff}$ . Since it is assumed that the missile can burn out at any position,  $\vec{R}$ , equation (5.2) can be written as

$$\vec{R}_T(t) = \vec{R} + \vec{V}_R t_{ff} + \frac{1}{2} \vec{g} t_{ff}^2 \quad (5.2A)$$

The target position is written as  $\vec{R}_T(t)$  to indicate that the position of the desired target may vary with time. The solution for  $\vec{V}_R$  on the flat earth is then

$$\vec{V}_R(\vec{R}, t) = \frac{\vec{R}_T(t) - \vec{R}}{t_{ff}} - \frac{1}{2} \vec{g} t_{ff} \quad (5.2B)$$

If the  $t_{ff}$  is now chosen as the fourth guidance restraint, then this pre-specified value (a constant) may be substituted in equation (5.2B) resulting in the desired expression for  $\vec{V}_R$ . If the fourth guidance restraint is that the total time of flight should be a prespecified constant,  $T$ , then  $t_{ff} = T - t$  should be substituted into equation (5.2B).

It may be required that the vertical velocity of the missile at the target be some prespecified value,  $\dot{Z}_T$ . Then, when  $g_x = g_y = 0$ ,

$$\dot{Z}_T = V_{Rz} + g t_{ff} \quad (g \approx -32 \frac{ft}{sec^2}) \quad (5.3)$$

When  $V_{Rz}$  is eliminated from equation (5.3) and the Z-component of equation (5.2B), an expression for time of free flight is obtained as

$$t_{ff} = \frac{\dot{Z}_T - \sqrt{\dot{Z}_T^2 - 2g(Z_T - Z)}}{g} \quad (5.4)$$

The desired expression for required velocity is now obtained by substituting  $t_{ff}$  from equation (5.4) into (5.2B).

In realistic cases  $\vec{V}_R$  must be obtained for oblate spheroids. This solution can be obtained with sufficient accuracy by analytical methods. However, these solutions will involve square roots and divisions which are difficult operations for many IGS computers. Experience indicates that a loss in accuracy results when these analytical solutions are expanded in power series expansions. Hence, curve fitting techniques (least squares) are employed using high-speed digital computers. It is desired to find  $\vec{V}_R(\vec{R}, t)$  in the following form:

$$\begin{aligned} V_{R_x} = & a_{00} + a_{10} \Delta X + a_{20} \Delta Y + a_{30} \Delta Z \\ & + a_{40} \Delta t + a_{11} \Delta X^2 + a_{12} \Delta X \Delta Y \\ & + \dots + a_{44} \Delta t^2 \end{aligned} \quad (5.5)$$

Similar expansions are obtained for  $V_{Ry}$  and  $V_{Rz}$ . There are 15 terms in each expansion. In equation (5.5),

$$\Delta X = X - X_N$$

$$\Delta Y = Y - Y_N$$

$$\Delta Z = Z - Z_N$$

$$\Delta t = t - t_N$$

$X_N, Y_N, Z_N, t_N$  = nominal (standard) burnout position and time

$a_{ij}$  = guidance constants, to be determined

The  $a_{ij}$  in equation (5.5) are analogous to partial derivatives, as

$$a_{10} = \left. \frac{\partial \dot{X}}{\partial X} \right)_{\text{Burnout}}$$

$$a_{12} = \left. \frac{\partial^2 \dot{X}}{\partial X \partial Y} \right)_{\text{Burnout}}$$

except that these "partial derivatives" are obtained by least squares techniques. Note that  $a_{00}$  in equation (5.5) is the nominal value of the X-component of the burnout velocity vector or

$$a_{00} = V_{RxN}$$

No third-order terms are given in equation (5.5). In general, the listed terms have been found to result in negligible fitting errors over the expected burnout dispersion box.

The burnout dispersion box is defined by the expected variations in  $\Delta X$ ,  $\Delta Y$ ,  $\Delta Z$ , and  $\Delta t$  at burnout. These variations result from variations in the thrust, specific impulse, initial mass, etc.

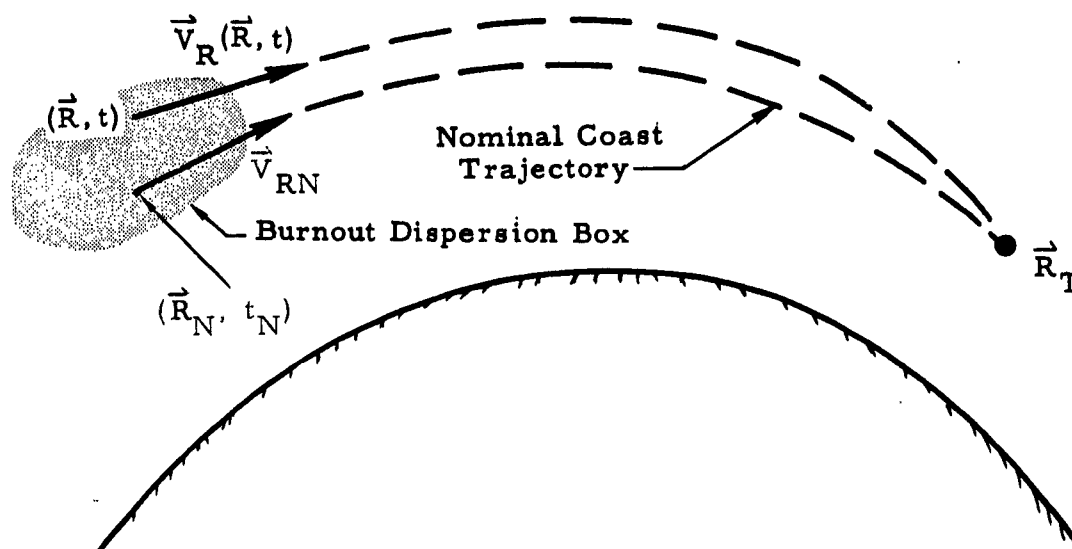


Figure 5.2 - Nominal and Nonstandard Required Velocity Vectors at Burnout.

The dispersion box is shown in Figure 5.2 along with nominal and nonstandard required velocity vectors at burnout. The size of the burnout dispersion box depends principally on the propulsion and control system tolerances. The shape depends on the guidance restraints chosen and the type of missile steering employed by the IGS.

In order to determine the  $a_{ij}$  in equation (5.5) by least squares procedures, it is necessary to have input data. To obtain these data, it is first necessary to estimate the size and shape of the dispersion box. When this has been done a large number of points may be designated within the dispersion box either at random or by some systematic procedure. Any one of many iterative techniques may now be used to determine the required velocity vectors at the designated points. With some techniques, as many as 15 machine runs may be required to determine  $\vec{V}_R$  at one



point. By definition, the resulting vector field,  $\vec{V}_R(\vec{R}, t)$  must satisfy the four chosen guidance constraints. This vector field is then the input data to the least squares routine.

A second method for obtaining these data has been used with success at STL. This method has the advantage that iterative techniques are not required. Assume that the four guidance restraints are

$$1, 2, 3) \vec{R} = \vec{R}_T$$

$$4) \dot{\vec{X}} = \dot{\vec{X}}_T$$

Then  $\dot{\vec{Y}}_T$  and  $\dot{\vec{Z}}_T$  (as well as the  $t_{ff}$ ) will vary. The free flight may be started at the target position with  $\dot{\vec{X}} = \dot{\vec{X}}_T$ . Runs may now be made, integrating backwards. These runs require that  $\dot{\vec{Y}}_T$  and  $\dot{\vec{Z}}_T$  be varied in amounts depending on the size of the expected dispersion box. When the missile has coasted (backwards) for the nominal time of free flight, the quantities  $\vec{R}$ ,  $\dot{\vec{R}}$ , and  $t_{ffN}$  are recorded. These data are also recorded, along the same free-flight trajectory at the times

$$t = t_{ffN} \pm n \Delta t \quad (n = 1, 2, \dots, N)$$

where  $\Delta t$  is also dependent on the size of the dispersion box. If  $N = 3$ , then seven sets of data, i. e., seven required velocity vectors, are obtained for each run. Then from the nominal and two variations on  $\dot{\vec{Y}}_T$  and  $\dot{\vec{Z}}_T$ , 35 sets of data will be obtained, which may be input into the least squares routine. Note that it is desirable to define all guidance restraints at the target, when using this method. If the velocity magnitude at the target were constrained, then  $\dot{\vec{X}}_T$  could be solved for and the above discussion applies. If the total time of flight,  $T$ , were constrained (as condition 4) it would be necessary to vary  $\Delta \dot{\vec{X}}_T$ , as well as  $\Delta \dot{\vec{Y}}_T$  and  $\Delta \dot{\vec{Z}}_T$ . The quantity  $\Delta t_{ff}$ , appearing in the expansion would then be eliminated using

$$T - t = t_{ff}$$

or 
$$-\Delta t = \Delta t_{ff} = t_{ff} - t_{ffN}$$

where 
$$\Delta t = t - t_N$$

Methods for steering the missile such that  $\dot{\vec{R}}$  will be controlled to be equal to  $\vec{V}_R(\vec{R}, t)$  will be presented in the next section. Note that such control is equivalent to controlling  $\vec{V}_g(\vec{R}, \dot{\vec{R}}, t)$  to zero, where

$$\vec{V}_g = \vec{V}_R - \dot{\vec{R}} \quad (5.6)$$

The quantity  $\vec{V}_g$  is used as the basic guidance control function in the MIT guidance system.

## VI. BALLISTIC MISSILE STEERING METHODS

A ballistic missile is steered by controlling the direction of the thrust vector. For flight through the atmosphere, the missile must be steered so that a gravity turn is approximated. A gravity turn results when the missile angle of attack, and hence aerodynamic forces normal to the missile axis, vanish. This requirement is made because normal forces of this nature set up bending moments along the missile axis that could cause the missile to break. It is shown in Appendix B that a gravity turn will result when  $\vec{\xi}$  is chosen as

$$\vec{\xi} \sim \dot{\vec{R}} - \vec{\omega}_E \wedge \vec{R} + \vec{W}^*$$

where  $\dot{\vec{R}}$  is the missile inertial velocity,  $\vec{\omega}_E$  is the earth angular velocity, and  $\vec{W}$  is the wind velocity. The wind velocity is usually neglected.

It will be assumed in this section that the commanded attitude turning rates required for the gravity turn are obtained from a simulation incorporating the above equation. The required turning rates may then be printed out and stored in a missile programmer. Alternate methods for accomplishing the gravity turn have been studied. For example, the required attitude may be expressed as an empirical function of actual missile position and velocity. This procedure minimizes the dispersions that result when closed-loop guidance steering is initiated.

The remaining paragraphs in this section will assume that the gravity turn has been completed and that the missile is out of the atmosphere. Hence, no aerodynamic forces will be considered to act on the missile.

Ballistic missiles generally have long, slender configurations with the thrust chamber gimbal located many feet away from the center of gravity (C.G.). Since the consideration of missile control systems would tend to mask the guidance problem, the following discussion will be limited to point mass missiles. Such missiles are approximated by cylindrical

---

\*The symbol  $\sim$  denotes parallelism.

shapes with the thrust chamber gimbal located at the C.G. The direction of the thrust attitude vector will be denoted by  $\vec{\xi}$  in this section. The IGS is also assumed to be located at the C.G. The purpose of the IGS is to determine a commanded value of the thrust attitude vector,  $\vec{\xi}_c$ , such that the missile velocity will be controlled to the required velocity vector.

Steering methods will be developed in this section for point mass missiles. These methods will be extended in Section VII to actual missiles, where the differential equations relating  $\vec{\xi}$  and  $\vec{\xi}_c$  will be obtained. It is sufficient for the present purpose to note that  $\vec{\xi}$  will lag  $\vec{\xi}_c$  as a result of control system reaction time, engine gimbal inertia, and any limiting that might be present. It is assumed here that the control system response is rapid and that these lags may be neglected. However, thrust misalignments are considered to exist, and integral control terms are added to reduce their effect. Hence,  $\vec{\xi}$  and  $\vec{\xi}_c$  will be treated as identical quantities in this section.

The problem of steering the missile may be approached in many ways. The solution to this problem is not unique. The methods presented in this section have been found to be satisfactory, but no claim is made that these methods are optimum. A short discussion of the nominal trajectory follows.

Trajectories for nominal missiles are usually generated by choosing a thrust attitude,  $\vec{\xi}_N(t)$ , such that

- a)  $\dot{\vec{R}} \rightarrow \vec{V}_R(\vec{R}, t)$
- b) Minimum propellant is wasted
- c) Heating constraints, etc., are satisfied
- d) Gravity turn is flown through the atmosphere

It may require many runs on a large digital computer of the powered-flight simulation to determine  $\vec{\xi}_N(t)$  subject to the listed restraints.

The nominal thrust attitude is determined from liftoff in the powered-flight simulation. Note that  $\vec{\xi}_N$  will change in direction except for vertical flight. It is customary to define a constant vector  $\vec{p}_N$  such that

$$\vec{\xi}_N(t) \cdot \vec{p}_N = 0$$

The plane defined by  $\vec{p}_N$  is said to be the pitch plane.

Since guided flight in a vacuum will be assumed, only two accelerations will act on the point mass missile. The sum of the linear accelerations

$$\begin{aligned} \ddot{\vec{R}} &= \vec{g} + \vec{a}_T \\ &= \vec{g} + a_T \vec{\xi} \end{aligned} \quad (6.1)$$

where  $\ddot{\vec{R}}$  = resultant inertial acceleration of the missile C.G.

$a_T$  = magnitude of thrust acceleration (= thrust per unit mass)

$\vec{\xi}$  = unit vector in the direction of the thrust vector

The ballistic missile must be steered such that  $\vec{R} \rightarrow \vec{V}_R(\vec{R}, t)$  at burnout for nominal as well as nonstandard missiles. The velocity to be gained,  $\vec{V}_g = \vec{V}_g(\vec{R}, \vec{R}, t)$  is defined as

$$\vec{V}_g = \vec{V}_R(\vec{R}, t) - \dot{\vec{R}} \quad (6.2)$$

Hence, a commanded thrust attitude vector,  $\vec{\xi}_c(t)$ , must be found such that all three components of  $\vec{V}_g$  vanish at burnout. At the precise time when  $\vec{V}_g = 0$  the missile engine will be cut off. The missile will then coast to its target satisfying the four guidance restraints.

It is convenient to choose the computing coordinate system set of basis vectors as follows:

$\vec{i}$  = parallel to the nominal thrust attitude vector at burnout

$$\vec{j} = \vec{p}_N, \text{ i.e., normal to the pitch plane}$$

$$\vec{k} = \vec{i} \wedge \vec{j}$$

The origin of the inertial coordinate system may be chosen at the center of the earth or at the launch site. The velocity of the missile will be

$$\dot{\vec{R}} = \dot{X} \vec{i} + \dot{Y} \vec{j} + \dot{Z} \vec{k}$$

Pitch commands may then be based on  $\dot{Z}_g$ , yaw commands may be based on  $\dot{Y}_g$ , and engine cutoff commands based on  $\dot{X}_g$ . These components are defined by equation (6.2), and  $\vec{V}_R$  is obtained by explicit computation or by using one of the methods given in Section V. It will be assumed that the change in thrust attitude is sufficiently small that cross-coupling between  $\dot{X}_g$ ,  $\dot{Y}_g$ , and  $\dot{Z}_g$  may be neglected.

When the thrust attitude,  $\vec{\xi}$ , is specified, the motion of the missile C.G. is obtained by integrating equation (6.1). Since  $\vec{\xi}$  is a unit vector the specification could be given in terms of the pitch and yaw components,  $\xi_z$  and  $\xi_y$ . The specification could also be given in terms of the pitch and yaw attitude rates,  $\dot{\xi}_z$  and  $\dot{\xi}_y$ . It is assumed that the commanded roll attitude rate is zero. The following paragraphs will discuss the determination of the pitch attitude. A corresponding discussion (not presented) will be obvious for the yaw channel.

The pitch attitude of the thrust vector,  $\xi_z$ , must be determined such that  $V_{gz} \rightarrow 0$ . This result would be obtained if  $V_{gz}$  were a solution of any one of the following six forms:

$$1) \quad -\ddot{Z} + KV_{gz} = 0$$

$$2) \quad \dot{V}_{gz} + KV_{gz} = 0$$

$$3)^* \quad -\ddot{Z} + KV_{gz} + L \int_{t_0}^t V_{gz} = 0$$

---

\*It is understood throughout that  $\int_{t_0}^t V_{gz} \equiv \int_{t_0}^t V_{gz}(\tau) d\tau$ .

$$4) \quad \dot{V}_{gz} + KV_{gz} + L \int_{t_0}^t V_{gz} = 0 \quad (6.4)$$

$$5) \quad -\ddot{Z} + K\dot{V}_{gz} + LV_{gz} = 0$$

$$6) \quad \ddot{V}_{gz} + K\dot{V}_{gz} + LV_{gz} = 0$$

where  $V_{gz} = V_{Rz} - \dot{Z}$ . Many other forms, including cross-coupled forms, could be added to the above list, but the above six will suffice for the present discussion. The gain functions  $K$  and  $L$  are chosen to insure stability and satisfactory transient response. The presence of the  $V_{gz}$  term in all six forms assures that  $V_{gz}$  will be controlled to zero. Forms (1) and (2) are the simplest, form (2) being preferred because  $\dot{V}_{gz}$  will also vanish at burnout. In practice, it is found that the presence of thrust misalignments, lags in the control system, etc., result in steering errors. These errors may be significantly reduced by introducing integral control terms as in forms (3) and (4). In these forms,  $t_0$  designates the time at which guidance is initiated. Forms (5) and (6) are equivalent to forms (3) and (4) and are used when the thrust attitude rate is to be commanded.

When form (4) is used,  $\xi_z$  is obtained by eliminating  $\ddot{Z}$  using equation (6.1) as

$$\ddot{Z} = g_z + a_T \xi_z \quad (6.5)$$

Then

$$a_T \xi_z = KV_{gz} + L \int_{t_0}^t V_{gz} + \dot{V}_{Rz} - g_z \quad (6.6)$$

Equation (6.6) may be simplified when  $\vec{V}_R(R, t)$  is chosen from a restricted class of vector functions. The restriction requires that whenever the missile engine is shut off, with  $\vec{V}_g = 0$ , that  $\vec{V}_g$  will continue to vanish during the coast period that precedes arrival at the target.

Guidance constraints of the type a) - c) on Page 21 will not result in expressions for  $\vec{V}_R(\vec{R}, t)$  that satisfy the class restriction. Constraints d) - h) will satisfy the restriction. In general, the class restriction is not a serious limitation. For example, whenever the four guidance restraints are defined at the target, the restriction will be satisfied. The remaining analysis in this report will assume that  $\vec{V}_R$  satisfies the class restriction.

When the guidance restraints are chosen such that  $\vec{V}_R(\vec{R}, t)$  satisfies the class restriction, it follows that

$$\vec{g} = \dot{\vec{V}}_R(\vec{R}, t), \quad \vec{V}_R = \dot{\vec{R}} \quad (6.7)$$

The following expansion in partial derivatives is valid in general

$$\dot{\vec{V}}_R = \sum_{a=1}^3 \frac{\partial \vec{V}_R}{\partial X^a} \dot{R}_a + \frac{\partial \vec{V}_R}{\partial t} \quad (6.8)$$

where

$$\dot{R}_1 = \dot{X}, \quad \dot{R}_2 = \dot{Y}, \quad \text{and} \quad \dot{R}_3 = \dot{Z}$$

When  $\vec{V}_R$  satisfies the class restriction, equations (6.7) may be substituted into (6.8) to obtain

$$\vec{g} = \sum_{a=1}^3 \frac{\partial \vec{V}_R}{\partial X^a} V_{Ra} + \frac{\partial \vec{V}_R}{\partial t} \quad (6.8A)$$

where

$$V_{R1} = V_{Rx}, \quad V_{R2} = V_{Ry}, \quad \text{and} \quad V_{R3} = V_{Rz}$$

Elimination of  $\frac{\partial \vec{V}_R}{\partial t}$  from equations (6.8A) and (6.8) gives

$$\vec{V}_R - \vec{g} = - \sum_{a=1}^3 \frac{\partial \vec{V}_R}{\partial X^a} V_{ga} \quad (6.9)$$



Using equation (6.9), equation (6.6) may be written as

$$a_T \xi_z = KV_{gz} + L \int_{t_0}^t V_{gz} - \sum_{a=1}^3 \frac{\partial V_{Rz}}{\partial X^a} V_{ga} \quad (6.10)$$

Note that equation (6.10) could be simplified if equation (6.4), form (4), had been written as

$$\dot{V}_{gz} + KV_{gz} + L \int_{t_0}^t V_{gz} - \sum_{a=1}^3 \frac{\partial V_{Rz}}{\partial X^a} V_{ga} = 0 \quad (6.11)$$

This simplification will not be made in this section because of the complex character of equation (6.11). However, it is felt that constants  $K$  and  $L$  for the pitch (and yaw) channels may be found that will cause  $V_{gz} \rightarrow 0$  and  $V_{gy} \rightarrow 0$ . Then the  $V_{ga}$  terms in equation (6.10) will not be required.

When a missile is steered according to equation (6.10) an excessive use of propellant may be experienced, since the nominal trajectory will not be reproduced. The nominal pitch thrust attitude is obtained as

$$\xi_{Nz} = \vec{\xi}_N \cdot \vec{k}$$

If  $\xi_z$  were evaluated along the nominal trajectory, it would generally be found that

$$\xi_z \neq \xi_{Nz}$$

By construction of  $\vec{\xi}_N$  it follows that any such deviations can only result in the failure of equation (6.10) to satisfy the nominal trajectory shaping restraints, with the exception of the first ( $\vec{R} \rightarrow \vec{V}_R$ ). This defect in equation (6.10) can be remedied in two ways, both essentially equivalent. A discussion of this procedure follows.

The reason why equation (6.10) causes the nominal missile to be steered (in pitch) is that equations (6.4) are not satisfied by the nominal missile. Equations (6.4), however, can be easily generalized so that they are satisfied by the nominal missile. This is done by adding a forcing function,  $F(V_{gx})$ , to the right-hand sides. The forcing function is obtained by evaluating the left-hand side of equation (6.4), form (4), over the nominal trajectory, to obtain  $F_z^N$  as

$$F_z^N = \dot{V}_{gzN} + K V_{gzN} + L \int_{t_0}^t V_{gzN} \quad (6.12)$$

The nominal value of the X-component of the velocity to be gained,  $V_{gxN}$ , is also obtained. Then  $F_z^N$  is expressed as a function of  $V_{gxN}$ . The quantity  $V_{gxN}$  is chosen as the independent variable for the F-function because  $V_{gx}$  will tend to zero in a smooth fashion as the missile gains velocity. At burnout  $V_{gx} = 0$ ; hence,  $V_{gz} \rightarrow 0$  as required. In order to retain computer simplicity,  $F_z^N$  is chosen in the form

$$F_z^N = C_1 V_{gxN} + C_2 V_{gxN}^2 \quad (6.13)$$

where  $C_1$  and  $C_2$  are trajectory shaping constants. These constants should be obtained such that a good fit to the nominal trajectory is obtained in the burnout region. The curve fit should also include the guidance initiation point in order to minimize pitch commands when the guidance loop is closed.

Although quadratic terms are indicated in equation (6.13), in some cases linear terms will suffice; in others cubic terms will be called for. This choice depends on the precision required in duplicating the nominal trajectory. Only a small amount of propellant waste is allowed for missions requiring maximum burnout energy. In such missions every additional X pounds of propellant required represents the loss of an additional physical experiment.

The forcing function,  $F_z(V_{gx})$ , to be added to the right-hand side of equation (6.4), form (4) is then

$$F_z(V_{gz}) = C_1 V_{gx} + C_2 V_{gx}^2 \quad (6.14)$$

where  $C_1$  and  $C_2$  are the same constants used in equation (6.13). Equation (6.4), form (4), is then modified as

$$\dot{V}_{gz} + K V_{gz} + L \int_{t_0}^t V_{gz} = F_z(V_{gx}) \quad (6.15)$$

Equation (6.15) is now satisfied by the nominal missile, and  $V_{gz} \rightarrow 0$  as required. The yaw and pitch attitudes to be commanded,  $\xi_y$  and  $\xi_z$ , may now be obtained following the procedure indicated by equations (6.5) to (6.10). The result is

$$a_T \xi_y = K V_{gy} + L \int_{t_0}^t V_{gy} - \sum_{a=1}^3 \frac{\partial V_{Rz}}{\partial X^a} V_{ga} - F_y(V_{gx}) \quad (6.15A)^*$$

$$a_T \xi_z = K V_{gz} + L \int_{t_0}^t V_{gz} - \sum_{a=1}^3 \frac{\partial V_{Rz}}{\partial X^a} V_{ga} - F_z(V_{gx})$$

where it is assumed that the pitch and yaw channels are identical; hence, the same gains  $K$  and  $L$ .

A second procedure will now be mentioned for shaping equations (6.4) so that these equations are satisfied by the nominal missile. Again let equation (6.4), form (4), be used as the example.

\* Note that equations (6.6), (6.10), (6.15A), and (6.18) can be simplified if the computing coordinate system is chosen such that

$$\left( \frac{\dot{V}_R}{V_R} - \vec{g} \right) \wedge \vec{i} = 0$$

The desired forcing function is

$$F_z^N = \dot{V}_{gzN} + K V_{gzN} + L \int_{t_0}^t V_{gzN}$$

Hence, form (4) could be written as

$$\dot{V}_{gz}^* + K V_{gz}^* + L \int_{t_0}^t V_{gz}^* = 0 \quad (6.16)$$

where

$$V_{gz}^* = V_{gz} - V_{gzN} \quad (6.17)$$

The quantity  $V_{gzN}$  may now be expressed in powers of  $V_{gxN}$  as was done in equation (6.13) for  $F_z^N$ . Note that this method requires that fitting errors in the integral and the derivative of  $V_{gzN}$  also be minimized. When this technique is used the expressions for  $\xi_y$  and  $\xi_z$  are obtained as

$$\begin{aligned} a_T \xi_y &= a_{Tn} \xi_{yn} + K V_{gy}^* + L \int_{t_0}^t V_{gy}^* - \sum_1^3 \frac{\partial V_{Ry}}{\partial X^a} V_{ga}^* \\ a_T \xi_z &= a_{Tn} \xi_{zn} + K V_{gz}^* + L \int_{t_0}^t V_{gz}^* - \sum_1^3 \frac{\partial V_{Rz}}{\partial X^a} V_{ga}^* \end{aligned} \quad (6.18)$$

where  $V_{g1}^* = V_{gx}^*$  and  $V_{g2}^* = V_{gy}^*$  are defined as in equation (6.17). That is

$$\vec{V}_g^* = \vec{V}_g - \vec{V}_{gN} \quad (6.19)$$

Since  $V_g^* \rightarrow 0$  at burnout, it follows that the variation between  $a_T$  and  $a_{TN}$  may be neglected. Then equations (6.18) become

$$\xi_y = \xi_{yN} + \frac{K}{a_T} V_{gy}^* + \frac{L}{a_T} \int_{t_0}^t V_{gy}^* - \frac{1}{a_T} \sum_1^3 \frac{\partial V_{Ry}}{\partial X^a} V_{ga}^* \quad (6.20)$$

$$\xi_z = \xi_{zN} + \frac{K}{a_T} V_{gz}^* + \frac{L}{a_T} \int_{t_0}^t V_{gz}^* - \frac{1}{a_T} \sum_1^3 \frac{\partial V_{Rz}}{\partial X^a} V_{ga}^*$$

Equations (6.20) completely determine  $\vec{\xi}_c$  since it is only necessary to specify two components of a unit vector, because equations (6.20) could also be written as

$$\begin{aligned} \psi &= \psi_N + \frac{K}{a_T} V_{gy}^* + \frac{L}{a_T} \int_{t_0}^t V_{gy}^* - \frac{1}{a_T} \sum_1^3 \frac{\partial V_{Ry}}{\partial X^a} V_{ga}^* \\ \theta &= \theta_N + \frac{K}{a_T} V_{gz}^* + \frac{L}{a_T} \int_{t_0}^t V_{gz}^* - \frac{1}{a_T} \sum_1^3 \frac{\partial V_{Rz}}{\partial X^a} V_{ga}^* \end{aligned} \quad (6.20A)$$

where  $\psi - \psi_N$  and  $\theta - \theta_N$  denote small variations in the pitch and yaw attitude angles about the nominal values.

Equations (6.15A) and (6.20) may now be used to steer the ballistic missile. The quantity  $V_{gx}$  will furnish the engine shutoff command. That is, when  $V_{gx} = \epsilon$  a command will be issued to shut off the main engine. The parameter  $\epsilon$  is included to allow for the expected residual impulse of the main engine, as well as the impulse of a vernier engine, if provided. The vernier engine will be shut off when  $V_{gx} = 0$ .

Since  $V_{gx}^* \approx 0$ , equations (6.20) can be written in vector form as

$$\vec{\xi} \sim \vec{\xi}_N + \frac{K}{a_T} \vec{V}_g^* + \frac{L}{a_T} \int_{t_0}^t \vec{V}_g^* - \frac{1}{a_T} \sum_1^3 \frac{\partial \vec{V}_R}{\partial X^a} \vec{V}_{ga}^* \quad (6.21)$$

where the gain constants  $K$  and  $L$  are also indicated for the roll channel. This is done simply for mathematical convenience and is acceptable since it is not planned to steer in roll. The symbol  $\sim$  is used in

equation (6.21) to denote parallelism. The X axis was chosen along  $\vec{\xi}_N$  at burnout. Hence, if  $\vec{\xi}_N$  is a constant vector, it follows that equation (6.21) is normalized except for second-order terms. When  $\vec{\xi}_N$  is not a constant vector, it is desirable to define the quantities  $\vec{v}_g^\dagger$  such that

$$\begin{aligned}\vec{v}_g^\dagger &= \vec{v}_g - \vec{\xi}_N \left( \vec{v}_g \cdot \vec{\xi}_N \right) \\ &= \vec{\xi}_N \wedge \left( \vec{v}_g \wedge \vec{\xi}_N \right)\end{aligned}\quad (6.22)$$

By construction,  $\vec{v}_g^\dagger$  is the sum of the two components of  $\vec{v}_g$  normal to the thrust attitude vector  $\vec{\xi}_N$ . The component of  $\vec{v}_g$  parallel to  $\vec{\xi}_N$  will be denoted by  $e$ , hence

$$e = \vec{v}_g \cdot \vec{\xi}_N \quad (6.23)$$

Then

$$\vec{v}_g = e \vec{\xi}_N + \vec{v}_g^\dagger$$

Both  $\vec{v}_g^*$  and  $\vec{v}_g^\dagger$  vanish along the nominal trajectory and are identical if  $\vec{\xi}_N$  is a constant. Hence, equations (6.21) may be written as

$$\dot{\vec{\xi}} = \vec{\xi}_N + \frac{K}{a_T} \vec{v}_g^\dagger + \frac{L}{a_T} \int_{t_0}^t \vec{v}_g^\dagger - \frac{1}{a_T} \sum_1^3 \frac{\partial \vec{v}_R}{\partial X^a} \vec{v}_{ga}^\dagger \quad (6.24)$$

Equation (6.24) may also be used for missile steering. The main engine cutoff signal is  $e$ , defined by equation (6.23). The advantage of this representation is that it is independent of a particular coordinate system and is free of cross-coupling errors.

The thrust attitude rate,  $\ddot{\vec{\xi}}$ , may be commanded in place of  $\dot{\vec{\xi}}$ . This expression may be derived by writing equation (6.4), form (6), as

$$\ddot{\vec{v}}_g + K \dot{\vec{v}}_g + L \vec{v}_g = \vec{q}(e) \quad (6.25)$$

The inhomogeneous term  $\vec{q}(\epsilon)$  is added so that equation (6.25) will be satisfied by the nominal trajectory. That is

$$\vec{q}(\epsilon_N) = \ddot{\vec{V}}_{gN} + K \dot{\vec{V}}_{gN} + L \vec{V}_{gN} \quad (6.26)$$

where  $\epsilon$  denotes the engine cutoff signal. Equations (6.1) and (6.2) may be differentiated with the result

$$\begin{aligned} \ddot{\vec{R}} &= \ddot{\vec{g}} + a_T \dot{\vec{\xi}} + \dot{a}_T \vec{\xi} \\ \ddot{\vec{V}}_g &= \ddot{\vec{V}}_R - \ddot{\vec{R}} \end{aligned}$$

Using the above results, equation (6.25) becomes

$$a_T \dot{\vec{\xi}} + \dot{a}_T \vec{\xi} = \left( \dot{\vec{V}}_R - \dot{\vec{g}} \right) + K \dot{\vec{V}}_g + L \vec{V}_g - \vec{q}(\epsilon) \quad (6.27)$$

The quantity  $\dot{a}_T \vec{\xi}$  may be absorbed in  $\vec{q}(\epsilon)$ . Then equation (6.27) will specify a commanded turning rate that will cause  $\vec{V}_g \rightarrow 0$ . Since,

$$\dot{\vec{V}}_g = \dot{\vec{V}}_R - \dot{\vec{g}} - a_T \vec{\xi}$$

equation (6.27) may be written as

$$a_T \dot{\vec{\xi}} + (\dot{a}_T + K a_T) \vec{\xi} = L \vec{V}_g + K (\dot{\vec{V}}_R - \dot{\vec{g}}) + (\dot{\vec{V}}_R - \dot{\vec{g}}) - \vec{q}(\epsilon) \quad (6.28)$$

The result of multiplication by  $\vec{\xi}$  (vector cross product) is

$$\begin{aligned} a_T \vec{\omega} &= L \vec{\xi} \wedge \vec{V}_g + \vec{\xi} \wedge \left( \dot{\vec{V}}_R - \dot{\vec{g}} \right) + \\ &\quad K \vec{\xi} \wedge \left( \dot{\vec{V}}_R - \dot{\vec{g}} \right) - \vec{\xi} \wedge \vec{q}(\epsilon) \end{aligned} \quad (6.29)$$

where  $\vec{\omega}$  is the commanded angular velocity of the thrust attitude vector defined as

$$\vec{\omega} = \dot{\vec{\xi}} \wedge \vec{\xi} \quad (6.30)$$

The steering equation (6.29) is one form of the so-called cross-product steering. This equation may be simplified using equations (6.9) and choosing the coordinate system suggested in the footnote on Page 37. Other simplifications will result when suitable cross-coupled terms are added to equation (6.4), form (6) as discussed in connection with equation (6.11). A stability analysis should then be done on the resulting form. Finally the steering equations should be simulated using a closed-loop guidance simulation. This last step is necessary to verify that only negligible errors ( $\vec{V}_g \neq 0$  at burnout) result when nonstandard missiles are steered.



## VII. BALLISTIC MISSILE CONTROL SYSTEMS

The preceding discussion was restricted to point mass missiles, where the thrust chamber was located at the C.G. Since this is not the case for long slender ballistic missiles, it is necessary to consider actual missile dynamics. Only rigid-body missile dynamics will be considered in this section. Aerodynamic forces will not be treated since it is assumed that the missile is above the atmosphere. The forces acting on the missile are shown in Figure 7.1. In Figure 7.1 the thrust attitude unit vector is denoted by the symbol  $\vec{\delta}$ . In the preceding section the thrust attitude was denoted by  $\vec{\xi}$ . This symbol will now be used to denote the missile attitude unit vector. When this change in notation is made, the linear acceleration equation becomes:

$$\ddot{\vec{R}} = \vec{g} + a_T \vec{\delta} \quad (7.1)$$

Two additional unit vectors are defined by the engine missile gimbal geometry. Pitch rotations of the missile occur about  $\vec{\zeta}$ ; yaw rotations about  $\vec{\eta}$ .

The relation between missile attitude,  $\vec{\xi}$ , and thrust attitude will now be found. This determination requires application of the turning moment equation. The moment equation for missiles is

$$\vec{\mathcal{T}} = I \dot{\vec{\omega}} \quad (7.2)$$

where

$\vec{\mathcal{T}}$  = torque about the missile C. G.

$I = I(t)$  is the missile pitch or yaw moment of inertia about the C. G.

$$\dot{\vec{\omega}} = \frac{d}{dt} \vec{\omega}$$

$$\vec{\omega} = \vec{\xi} \wedge \dot{\vec{\xi}}$$

Equation (7.2) neglects propellant sloshing effects; it is assumed that all internal linear momenta are directed along the missile axis. In order for equation (7.2) to have the given simple form, the roll angular velocity must vanish. Then the roll moment of inertia is not involved. The torque

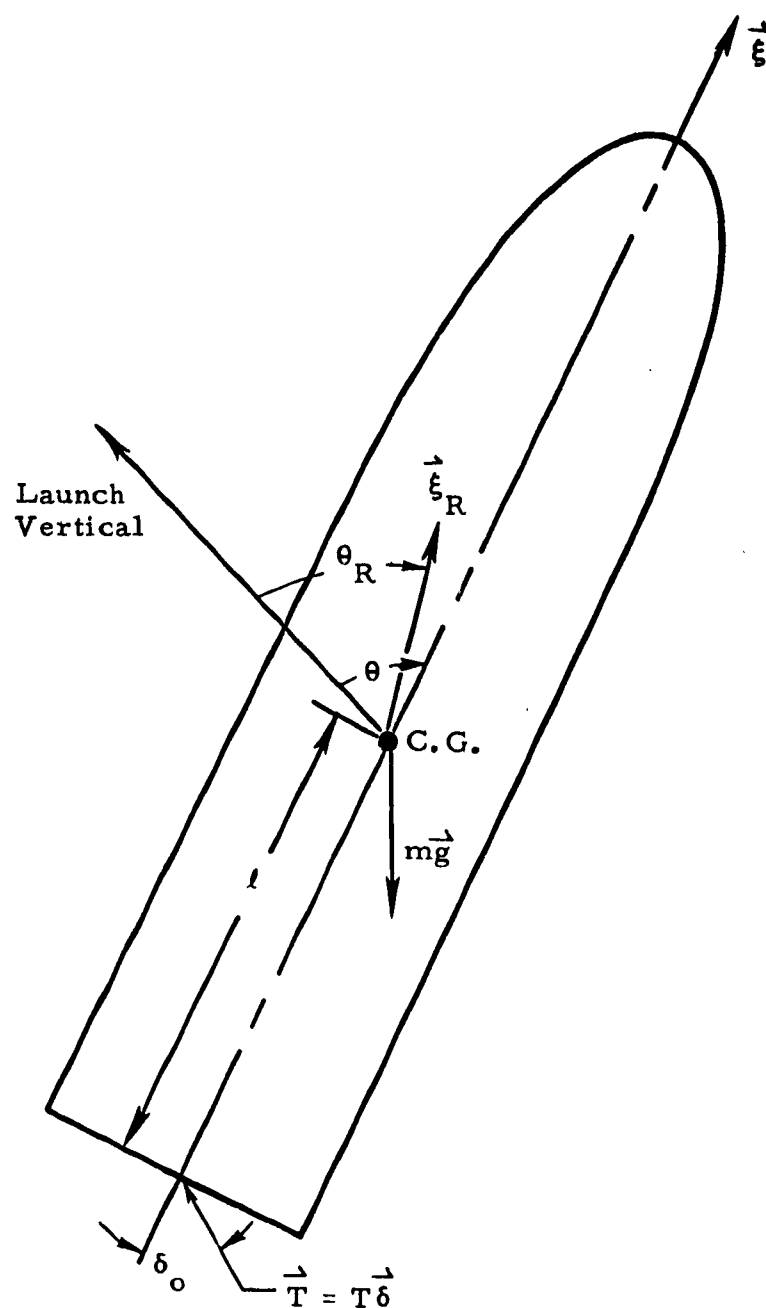


Figure 7.1 - Forces Acting on the Ballistic Missile During Powered Flight in a Vacuum.

about the missile C.G. is

$$\vec{J} = \vec{T} \wedge (l \vec{\xi}) = l T \vec{\delta} \wedge \vec{\xi} \quad (7.3)$$

Hence,

$$\mu_c \vec{\delta} \wedge \vec{\xi} = \vec{\omega} \quad (7.4)$$

where

$$\mu_c = \frac{T l}{Y} \quad (7.5)$$

Equation (7.4) is usually presented in component form. The pitch component is obtained by multiplying (scalar product) equation (7.4) by  $\vec{\zeta}$  to obtain

$$\mu_c \delta_o \approx \ddot{\theta}$$

where

$$\begin{aligned} \delta_o &= \text{pitch angle between } \vec{\delta} \text{ and } \vec{\xi} \left( \approx \vec{\delta} \wedge \vec{\xi} \cdot \vec{\zeta} \right) \\ \ddot{\theta} &= \text{magnitude of pitch angular acceleration} \left( \approx \vec{\omega} \cdot \vec{\zeta} \right) \end{aligned}$$

The symbol  $\approx$  is used above to indicate that small angle approximations have been made.

The purpose of the missile control system is to steer the thrust vector,  $\vec{T}$ , such that the missile will assume a reference attitude,  $\vec{\xi}_R$ . The reference attitude is chosen to coincide with the desired thrust attitude determined in the preceding section. The missile moment equation (7.4) indicates the validity of this identification. When  $\vec{\xi} \neq \vec{\delta}$ , the missile must experience an angular acceleration,  $\vec{\omega}$ . The assumption that  $\vec{\omega}$  is negligible is acceptable when guidance steering commands only are considered. That is,  $\vec{\omega}$  is large as a result of control system commands rather than guidance system commands. When  $\vec{\omega} = 0$ , equation (7.1) reduces to

$$\ddot{\vec{R}} = \vec{g} + a_T \vec{\xi}$$

This is precisely the same equation that was discussed in Section VI.

Expressions for the reference attitude were given in equations (6.15), (6.18), (6.20), (6.21), and (6.24). The latter is

$$\vec{\xi}_R = \vec{\xi}_N + \frac{K}{a_T} \frac{\vec{V}}{g} + \frac{L}{a_T} \int_{t_0}^t \frac{\vec{V}}{g} - \frac{1}{a_T} \sum_1^3 \frac{\partial \vec{V}_R}{\partial X^a} \frac{\vec{V}}{g^a}$$

The commanded thrust attitude,  $\vec{\delta}_c$ , must now be determined to insure that the missile attitude will be controlled to  $\vec{\xi}_R$ .

The commanded pitch or yaw thrust attitude is usually presented in component form, the pitch component (typical) is

$$\delta_{oc} = K_D (\theta_R - \theta) - K_R \dot{\theta} \quad (7.7)$$

where

$\delta_{oc}$  = pitch component of commanded thrust attitude ( $\approx \delta_c \cdot \vec{\zeta}$ )

$K_D, K_R$  = control system constants

$\theta_R - \theta$  = pitch angle between  $\vec{\xi}_R$  and  $\vec{\xi}$  ( $\approx \vec{\xi} \wedge \vec{\xi}_R \cdot \vec{\zeta}$ )

$\dot{\theta}$  = magnitude of the pitch angular velocity ( $\approx \dot{\omega} \cdot \vec{\zeta}$ )

In equation (7.7) the quantity  $\dot{\theta}$  is usually measured by a rate gyro package located at some point in the missile structure that is relatively free of bending frequencies.

The control system equations may be specified in a form that is independent of any particular coordinate system. Equation (7.7) is readily

generalized to the vector form. The result is

$$\vec{\delta}_c \wedge \vec{\xi} \approx K_D \vec{\xi} \wedge \vec{\xi}_R - K_R \vec{\omega} \quad (7.8)$$

Equation (7.8) defines both pitch and yaw engine deflection commands. The same constants  $K_D$  and  $K_R$  are used for both pitch and yaw since these channels are assumed to be identical. In this section, as in the preceding section, the lag in the engine hydraulic system will be neglected. Hence, the quantities  $\vec{\delta} = \vec{\delta}_c$  will be considered to be identical. When  $\vec{\delta} = \vec{\delta}_c$  is eliminated from equations (7.8) and (7.4) the result is

$$\ddot{\vec{\omega}} + \mu_c K_R \dot{\vec{\omega}} - \mu_c K_D \vec{\xi} \wedge \vec{\xi}_R \approx 0 \quad (7.9)$$

The pitch component of equation (7.9) is obtained by multiplying (scalar product) equation (7.9) by  $\vec{\xi}$ . The result is

$$\ddot{\theta} + \mu_c K_R \dot{\theta} + \mu_c K_D (\theta - \theta_R) \approx 0 \quad (7.10)$$

Equation (7.10) is usually obtained by eliminating  $\delta_o = \delta_{oc}$  between equations (7.7) and (7.6). A consideration of equation (7.10) serves to justify the form chosen for  $\vec{\delta}_c$  in equations (7.7) or (7.8). The control system constants  $K_D$  and  $K_R$  must be chosen such that a stable system with satisfactory transient response is obtained. In practice, missile bending and propellant sloshing must be considered so that a simulation is required for this gain determination. Values of  $K_D$  and  $K_R$  may generally be found such that  $\vec{\xi} \rightarrow \vec{\xi}_R$  with small lag.

If some additional computer complexity is allowed, then  $\vec{\xi}_R$  can be calculated as in equation (6.27). The pitch component (typical) of  $\vec{\delta}_c$

could then be written in the form:

$$\delta_{oc} = K_D(\theta_R - \theta) + K_R(\dot{\theta}_R - \dot{\theta})$$

Equations of this type would control  $\vec{\xi}$  to  $\vec{\xi}_R$  as well as  $\dot{\vec{\xi}}$  to  $\dot{\vec{\xi}}_R$ , a desirable feature when the nominal missile is pitched over at a high rate.

Some additional relations, useful in guidance and control systems analysis, will now be given. Since  $\vec{\omega} = \vec{\xi} \wedge \dot{\vec{\xi}}$ , equation (7.8) may be written as:

$$\vec{\delta}_c \wedge \vec{\xi} = K_D \vec{\xi} \wedge \dot{\vec{\xi}}_R - K_R \dot{\vec{\xi}} \wedge \vec{\xi} \quad (7.11)$$

Equation (7.11) will now be assumed to define  $\vec{\delta}_c$ , hence, the omission of the  $\approx$  symbol. An equivalent form of equation (7.11) is

$$\vec{\xi} \wedge \left[ \vec{\delta}_c + K_D \dot{\vec{\xi}}_R - K_R \dot{\vec{\xi}} \right] = 0 \quad (7.12)$$

The quantity in the square parenthesis must be parallel to  $\vec{\xi}$ , hence,

$$A \vec{\xi} = \vec{\delta}_c + K_D \dot{\vec{\xi}}_R - K_R \dot{\vec{\xi}} \quad (7.13)$$

The normalization constant, A, may be determined by multiplying (scalar product) both sides of equation (7.9) by  $\vec{\xi}$ . Then

$$A = \vec{\delta}_c \cdot \vec{\xi} + K_D \dot{\vec{\xi}}_R \cdot \vec{\xi} \quad (7.14)$$

The result of substituting A into equation (7.13) is

$$\vec{\delta}_c = \vec{\xi} (\vec{\delta}_c \cdot \vec{\xi}) - K_D \left[ \dot{\vec{\xi}}_R - \vec{\xi} (\dot{\vec{\xi}}_R \cdot \vec{\xi}) \right] + K_R \dot{\vec{\xi}} \quad (7.15)$$

When the small angle approximation is made, equation (7.15) becomes

$$\vec{\delta}_c \approx \vec{\xi} - K_D (\dot{\vec{\xi}}_R - \dot{\vec{\xi}}) + K_R \dot{\vec{\xi}} \quad (7.16)$$

Equation (7.16) gives a simple relation between  $\vec{\delta}_c$ ,  $\vec{\xi}$ , and  $\vec{\xi}_R$ , that does not depend on any particular inertial coordinate system.

A vector relation may also be found relating  $\vec{\delta}$  and  $\vec{\xi}$ . Equation (7.4) is

$$\mu_c \vec{\delta} \wedge \vec{\xi} = \frac{d}{dt} (\vec{\xi} \wedge \vec{\xi}) \quad (7.17)$$

Since  $\vec{\xi} \wedge \vec{\xi} = 0$ , the moment equation reduces to

$$\vec{\xi} \wedge \left[ \mu_c \vec{\delta} + \frac{d}{dt} \vec{\xi} \right] = 0 \quad (7.18)$$

It may be shown that

$$\frac{d}{dt} \vec{\xi} = -\omega^2 \vec{\xi} + \dot{\omega} \frac{\vec{\xi}}{\omega} \quad (7.19)$$

where  $\omega$  and  $\dot{\omega}$  are the magnitudes of  $\vec{\omega}$  and  $\dot{\vec{\omega}}$ . With this substitution, equation (7.18) becomes

$$\vec{\xi} \wedge \left[ \mu_c \vec{\delta} + \dot{\omega} \frac{\vec{\xi}}{\omega} \right] = 0 \quad (7.20)$$

An expression for  $\vec{\delta}$  may be obtained from equation (7.20) using the method of the preceding paragraph. The result is

$$\vec{\delta} = \vec{\xi} (\vec{\delta} \cdot \vec{\xi}) - \frac{\dot{\omega}}{\mu_c \omega} \vec{\xi} \quad (7.21)$$

Since  $\vec{\xi}$  and  $\frac{\dot{\vec{\xi}}}{\omega}$  are unit vectors, it follows that

$$\begin{aligned}\cos \delta_1 &= \vec{\delta} \cdot \vec{\xi} \\ \sin \delta_1 &= \frac{\dot{\omega}}{\mu_c}\end{aligned}\quad (7.22)$$

where  $\delta_1$  is the angle (combined pitch and yaw) between  $\vec{\delta}$  and  $\vec{\xi}$ . When the small angle approximation is made, equation (7.21) becomes

$$\vec{\delta} \approx \vec{\xi} - \frac{\dot{\omega}}{\mu_c} \frac{\dot{\vec{\xi}}}{\omega} \quad (7.23)$$

The above discussion presented the basic equations of a missile control system necessary for guidance system analysis. The discussion assumed that the missile attitude was to be controlled to the guidance reference attitude  $\vec{\xi}_R$ . Only simple modifications are required to adapt the analysis to the case when  $\vec{\xi}$  is to be controlled to  $\vec{\xi}_R$  or  $\vec{\omega}$  to  $\vec{\omega}_R$ .

A small complication could be added to the above analysis by including the effect of the engine gimbal inertia. The relationship between  $\vec{\delta}$  and  $\vec{\delta}_c$  is used in this section as  $\vec{\delta} = \vec{\delta}_c$ . A better approximation for this relation is

$$\delta_o(t) = k \int_0^t e^{-k\tau} \delta_{oc}(t - \tau) d\tau \quad (7.24)$$

for the pitch (typical) component. In equation (7.24) the parameter  $k$  is related to the lag in the engine hydraulic system.



## REFERENCES

- (1) Moore, R.A. and D.F. Meronek. "A Digital Computer Program for a Generalized Inertial Guidance System Error Analysis." Space Technology Laboratories, Inc., STL/TR59-0000-00647, 30 March 1959. (U)
- (2) Moore, R.A. "The Evaluation of Missile Accuracy." Space Technology Laboratories, Inc., STL/TR-59-0000-00678, 18 May 1959. (U)
- (3) Russell, W.T. "Inertial Guidance for Ballistic Vehicles." Unpublished paper, dated 15 October 1958.
- (4) Grubin, Carl. "Three Dimensional Rotational Dynamics of a Single-Axis Floated, Integrated Gyroscope." Space Technology Laboratories, Inc., STL/TN-59-0000-00406, December 1959. (U)
- (5) Whitcombe, D.W. "Polynomial Gravity Expansion for ARMA Inertial Guidance System." Ramo-Wooldridge Corporation, GM-TN-43, 17 December 1956. (C)

## APPENDIX A

### Derivation of the Force Equation for Ballistic Missiles

A short nonrigorous derivation of the linear force equation is given in this appendix. The sum of the external forces acting on a system of particles equals the rate of change of linear momentum of the system or

$$\sum \vec{F} = (m\dot{\vec{v}} + m_g \dot{\vec{v}}_g) \quad (\text{A-1})$$

where

$m$  = mass of the missile

$\vec{v}$  = velocity of the missile

$m_g$  = mass of the escaping gas

$\vec{v}_g$  = velocity of the escaping gas

and the dot is used to denote differentiation with respect to time. It is assumed in this appendix that the only external force on the missile arises from gravitational acceleration; hence, equation (A-1) may be written as

$$m\dot{\vec{g}} = m\dot{\vec{v}} + \dot{m}\vec{v} + m\dot{\vec{v}}_g + \dot{m}_g\vec{v}_g \quad (\text{A-2})$$

Now  $\dot{\vec{v}}_g = 0$  when the gas exits into free space, and  $\dot{m} = -\dot{m}_g$  since the total system mass is a constant. Thus

$$\begin{aligned} m\dot{\vec{g}} &= m\dot{\vec{v}} + \dot{m}_g(\vec{v}_g - \vec{v}) \\ &= m\dot{\vec{v}} + \dot{m}_g\vec{C} \\ &= m\dot{\vec{v}} - \vec{T} \end{aligned} \quad (\text{A-3})$$

where

$\vec{C}$  = escape gas velocity with respect to the missile  
(also called specific impulse)

$\vec{T} = -\dot{m}_g \vec{C}$  = missile thrust vector.

Equation (A-3) may be divided by  $m$ , with the result

$$\ddot{\vec{R}} = \dot{\vec{v}} = \vec{g} + \vec{a}_T \quad (\text{B-4})$$

where

$$\vec{a}_T = \frac{\vec{T}}{m}$$

## 7

## The Gravity Turn Pitch Program

The missile is steered through the atmosphere such that a "gravity turn" is followed. This pitch profile is alternately referred to as either a zero-lift or a zero angle-of-attack pitch program. This program is utilized in order to prevent breakage of the missile as a result of aerodynamic forces.

The aerodynamic forces are referred to as drag,  $\vec{D}$ , and lift,  $\vec{L}$ . The drag force is directed along the roll axis while the lift force acts normal to the missile axis. The forces acting on the missile, for flight through the atmosphere, are shown in Figure B-1. The point of application of the resultant aerodynamic force is referred to as the center of pressure (C.P.). The forces,  $\vec{L}$  and  $\vec{D}$ , acting at the C.P. are defined as:

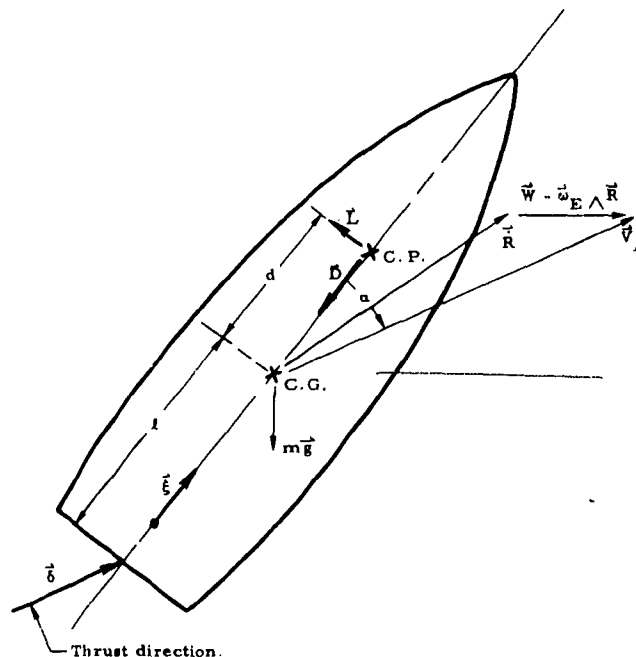


Figure B-1 - Forces Acting on the Missile for Flight Through the Atmosphere

$$\vec{D} = -D \vec{\xi} \quad (B-1)$$

$$\vec{L} = \frac{L}{V_A} \vec{\xi} \wedge (\vec{V}_A \wedge \vec{\xi}) \quad (B-2)$$

where:

$$D = C_D A q$$

$$L = C_N(a) A q$$

$$\vec{\xi} = \text{direction of missile roll axis, a unit vector}$$

$$\vec{V}_A = \text{velocity of the missile with respect to the air mass}$$

$$V_A = \text{magnitude of } V_A$$

$$A = \text{effective missile area}$$

$$q = 1/2 \rho V_A^2 = \text{dynamic pressure}$$

$$\rho = \text{air density}$$

$$C_D = \text{drag coefficient}$$

$$C_N(a) = \text{lift coefficient}$$

$$a = \text{angle of attack } (V_A \cos a = \vec{V}_A \cdot \vec{\xi})$$

The drag and lift forces are sometimes defined as acting along and normal to  $\vec{V}_A$ , rather than  $\vec{\xi}$ . When actual data are used, care must be taken to insure that the definitions are consistent with the data.

The axial strength of the missile is greater than the transverse strength. Hence, the normal forces (lift) must be minimized for flight through the atmosphere. Otherwise the aerodynamic lift forces would produce bending moments that could break the long, slender missile.

Note also that when  $\vec{L} \neq 0$ , it is necessary to choose the engine thrust direction,  $\vec{\delta}$ , to prevent the missile from rotating. The aerodynamic pitching moment is cancelled by choosing

$$\begin{aligned}\vec{L}_d &= \vec{\xi} \wedge (\vec{\delta} \wedge \vec{\xi}) l T \\ &= \left[ \vec{\delta} - \vec{\xi} (\vec{\xi} \cdot \vec{\delta}) \right] T l\end{aligned}$$

where

$d$  = distance between the C.G. and the C.P.

$l$  = distance between the C.G. and the engine gimbal

$T$  = engine thrust

When a zero-lift pitch program is not followed the energy required to cancel the pitching moment is wasted. Some of this wasted energy is converted to heat energy resulting in weakening of the missile structure.

Note that  $\vec{L}$  can be made to vanish by choosing

$$\vec{\xi} = \frac{\vec{V}_A}{V_A} \quad (B-3)$$

The quantity  $\vec{V}_A$  is calculated as

$$\vec{V}_A = \vec{R} - \vec{\omega}_E \wedge \vec{R} + \vec{W} \quad (B-4)$$

where

$\vec{R}$  = velocity of the missile with respect to inertial space

$\vec{R}$  = position vector of the missile with respect to the earth

$\vec{\omega}_E$  = angular velocity of the earth

$\vec{W}$  = wind velocity with respect to the earth, normally a negligible quantity

The missile will fly a gravity turn when equation (B-3) is satisfied. All the quantities required to determine the direction of  $\vec{V}_A$  are calculated in standard missile simulations. Then the thrust attitude,  $\vec{\delta}$ , may be commanded, as

$$\vec{\delta} = \frac{\vec{V}_A}{V_A}$$

When a unity control system (equivalent to a point mass missile) is assumed, it follows that  $\vec{\delta} = \vec{\xi}$ .

The simulation will calculate the missile angular velocity,  $\vec{\omega} = \vec{\xi} \wedge \dot{\vec{\xi}}$ . This quantity may be approximated with simple functions and incorporated in a missile pitch programmer. Steering commands from the missile programmer then cause the missile to pitch over to approximate the desired gravity turn.

Note that when  $\vec{W} \approx 0$ , the initial value of  $\vec{\xi} = \vec{\delta}$  is arbitrary. When the initial value,  $\vec{\xi} = \vec{\xi}_0$ , is chosen the resulting gravity turn is specified. In general it is necessary to make several flights, using different values for  $\vec{\xi}_0$ , in order to obtain the desired end conditions. These variations can be made even when  $\vec{W} \neq 0$ , since wind velocities are normally small enough to be neglected.

UNCLASSIFIED

Aerospace Corporation, El Segundo, California.  
AN INTRODUCTION TO INERTIAL GUIDANCE CON-  
CEPTS FOR BALLISTIC MISSILES (A TUTORIAL  
REPORT), by D. W. Whitcombe. 15 March 1961.  
60p. incl. illus. (Report No. TDR-594(1990)TN-1)  
(Contract AF 04(647)-594).

Unclassified report

Various aspects of guidance and control for ballistic missiles are presented. A discussion of the required velocity vector is given with several examples based upon the flat earth model. Error analysis procedures involving inertial reference and measurement systems are treated in some detail.

Missile steering techniques are approached in such a way that stability is not a problem. Procedures for insuring that the missile velocity will be controlled to the required velocity without payload loss penalties are given. This is accomplished by introducing trajectory-shaping functions that cause the guided (over)

UNCLASSIFIED

UNCLASSIFIED

Aerospace Corporation, El Segundo, California.  
AN INTRODUCTION TO INERTIAL GUIDANCE CON-  
CEPTS FOR BALLISTIC MISSILES (A TUTORIAL  
REPORT), by D. W. Whitcombe. 15 March 1961.  
60p. incl. illus. (Report No. TDR-594(1990)TN-1)  
(Contract AF 04(647)-594).

Unclassified report

Various aspects of guidance and control for ballistic missiles are presented. A discussion of the required velocity vector is given with several examples based upon the flat earth model. Error analysis procedures involving inertial reference and measurement systems are treated in some detail.

Missile steering techniques are approached in such a way that stability is not a problem. Procedures for insuring that the missile velocity will be controlled to the required velocity without payload loss penalties are given. This is accomplished by introducing trajectory-shaping functions that cause the guided (over)

UNCLASSIFIED

UNCLASSIFIED

Aerospace Corporation, El Segundo, California.  
AN INTRODUCTION TO INERTIAL GUIDANCE CON-  
CEPTS FOR BALLISTIC MISSILES (A TUTORIAL  
REPORT), by D. W. Whitcombe. 15 March 1961.  
60p. incl. illus. (Report No. TDR-594(1990)TN-1)  
(Contract AF 04(647)-594).

Unclassified report

Various aspects of guidance and control for ballistic missiles are presented. A discussion of the required velocity vector is given with several examples based upon the flat earth model. Error analysis procedures involving inertial reference and measurement systems are treated in some detail.

Missile steering techniques are approached in such a way that stability is not a problem. Procedures for insuring that the missile velocity will be controlled to the required velocity without payload loss penalties are given. This is accomplished by introducing trajectory-shaping functions that cause the guided (over)

UNCLASSIFIED

UNCLASSIFIED

Aerospace Corporation, El Segundo, California.  
AN INTRODUCTION TO INERTIAL GUIDANCE CON-  
CEPTS FOR BALLISTIC MISSILES (A TUTORIAL  
REPORT), by D. W. Whitcombe. 15 March 1961.  
60p. incl. illus. (Report No. TDR-594(1990)TN-1)  
(Contract AF 04(647)-594).

Unclassified report

Various aspects of guidance and control for ballistic missiles are presented. A discussion of the required velocity vector is given with several examples based upon the flat earth model. Error analysis procedures involving inertial reference and measurement systems are treated in some detail.

Missile steering techniques are approached in such a way that stability is not a problem. Procedures for insuring that the missile velocity will be controlled to the required velocity without payload loss penalties are given. This is accomplished by introducing trajectory-shaping functions that cause the guided (over)

UNCLASSIFIED



UNCLASSIFIED	<p>ballistic missile to closely follow the nominal reference trajectory.</p> <p>The steering discussion initially assumes a point mass missile. Control system complications that result from actual missile dynamics are treated separately.</p>	UNCLASSIFIED
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED
UNCLASSIFIED	<p>ballistic missile to closely follow the nominal reference trajectory.</p> <p>The steering discussion initially assumes a point mass missile. Control system complications that result from actual missile dynamics are treated separately.</p>	UNCLASSIFIED